An Intriguing ODE
Graduate Seminar, Spring 2021

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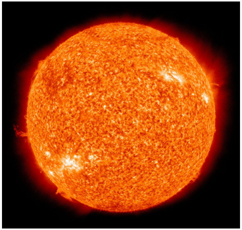
Mathematics & Statistics
UNC Wilmington



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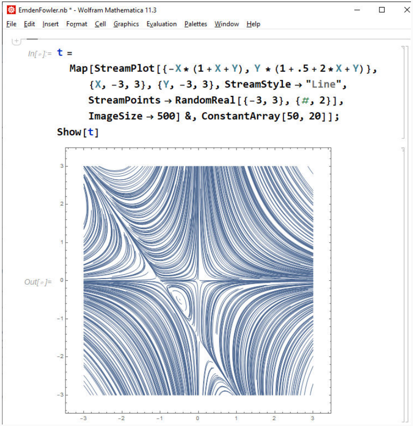


Figure 1: Nonlinear dynamics in Mathematica.

In Search of a Problem - Down the Rabbit Hole of Sources

- MAT 495 students needed a research topic.
- They had no idea what they wanted to do.
- Interested in applications, physics, nonlinear dynamics, solar energy.
- A search led to a paper: Mancas and Rosu (2015), *Existence of periodic orbits in nonlinear oscillators of Emden-Fowler form*, **Phys. Lett. A 380**, 422-428.
- The paper referred to applications:
    - Gravitational potential of self-gravitating gas.
    - Used by Eddington - internal constitution of stars.
    - Thomas-Fermi model of electrons in atoms.
    - Authors studied a related dynamical system.
    - Generalization of Lane-Emden equation.

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n.$$

Searching the Literature and History

- Jonathan Homer Lane (1819-1880)
  1870 - *On the Theoretical Temperature of the Sun under the Hypothesis of a Gaseous Mass maintaining its Volume by its Internal Heat, and depending on the Laws of Gases as known to Terrestrial Experiment.*
- (Jacob) Robert Emden (1862-1940)
  1907 *Gaskugeln* - thermal behavior of a spherical cloud of gas acting under mutual attraction of molecules.
- Ralph Howard Fowler (1889-1944)
  Published papers in 1914, 1930, 1931 on a generalized equation.
- Subrahmanyan Chandrasekhar (1910-1995)
  An Introduction to the Study of Stellar Structure, 1939.

Stellar Structure - Understanding the Background

- Hydrostatic Equilibrium

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}.$$

- Mass Conservation

$$\frac{dM}{dr} = -4\pi\rho(r)r^2.$$

- Then,

$$\begin{aligned} \frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP}{dr}\right) &= -G\frac{dM(r)}{dr} \\ &= -4\pi G\rho(r)r^2. \end{aligned}$$

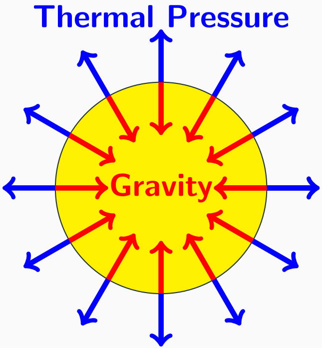


Figure 2: Balance of thermal pressure with gravitational pressure. (Done in LaTeX.)

Polytropic Models

- So far, we have

$$\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP}{dr}\right) = -4\pi G\rho(r)r^2.$$

- Need an Equation of State, $P = P(\rho)$.

$$P = K\rho^\gamma, \quad \gamma = \frac{n+1}{n}.$$

$n =$ polytropic index.

  - $n = 0$, rocky planets.
  - $0.5 \leq n \leq 1$, neutron stars.
  - $n = 3$, white dwarfs, Sun.

- Gives Poisson Equation: $\nabla^2\rho = f$.

$$\frac{(n+1)K}{4\pi nG}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{\frac{n-1}{n}}}\frac{d\rho}{dr}\right) = -\rho.$$

Lane-Emden Equation

- Rewrite the Poisson Equation:

$$\frac{(n+1)K}{4\pi nG}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{\frac{n-1}{n}}}\frac{d\rho}{dr}\right) = -\rho.$$

- Let $\rho = \rho_c\theta^n$, $r = \alpha\xi$, then

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n.$$

$n = 3$	Eddington
$n = \frac{3}{2}$	Thomas-Fermi Model

- Initial Conditions,
  $\theta(0) = 1, \theta'(0) = 0.$
- Fowler generalized to

$$\frac{d}{d\xi}\left(\xi^\rho\frac{d\theta}{d\xi}\right) + b\xi^\sigma\theta^n = 0.$$

- Chandrasekhar - white dwarfs

Known Solutions: $\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n,\ \theta(0)=1,\ \theta'(0)=0.$

- For $n=0$, $\theta'' + \dfrac{2}{\xi}\theta' + 1 = 0$. Use integrating factor, ξ^2.

$$\begin{aligned} \left(\xi^2\theta'\right)' &= -\xi^2 \\ \xi^2\theta' &= -\frac{1}{3}\xi^3 + c_1 \\ \theta' &= -\frac{1}{3}\xi + \frac{c_1}{\xi^2} \\ \theta &= -\frac{1}{6}\xi^2 - \frac{c_1}{\xi} + c_2. \end{aligned}$$

- For $n=1$, $\theta'' + \dfrac{2}{\xi}\theta' + \theta = 0$. **Solution:** $\theta = \dfrac{c_1\cos\xi + c_2\sin\xi}{\xi}$.
- Apply conditions: $n=0 : \theta = 1 - \dfrac{\xi^2}{6}$, $n=1 : \theta = \dfrac{\sin\xi}{\xi}$.
- $n=5$, $\theta = \dfrac{1}{\sqrt{1+\frac{1}{3}\xi^2}}$.

Numerical Solutions

Write $\theta'' = -\frac{2}{\xi}\frac{d\theta}{d\xi} - \theta^n$.

For $\theta_i \approx \theta(\xi_i)$, $\xi_i = i\Delta\xi, i = 0, 1, \ldots, N$, approximate θ' by

$$\theta_{i+1} = \theta_i + \Delta\xi \frac{d\theta}{d\xi}$$

and approximate θ'' by

$$\left(\frac{d\theta}{d\xi}\right)_{i+1} = \left(\frac{d\theta}{d\xi}\right)_i + \Delta\xi \left[\frac{2}{\xi}_i \left(\frac{d\theta}{d\xi}\right)_i - \theta_i^n\right].$$

Set conditions $\theta_0 = 1$, $\theta_1 = \theta_0$.

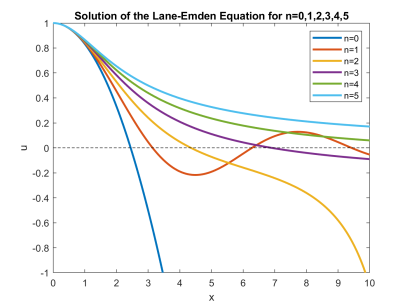


Figure 3: Numerical Solutions for $n = 0, 1, 2, 3, 4, 5$. **Note** - Zeros are important.

MATLAB Code for Emden Fowler: $\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = \alpha\xi^\lambda\theta^n.$

```
clear
lambda=2;
N=50000;
a=0; b=10; dt=(b-a)/N;
for j=1:6          % Get n values
    n=j-1;
    x=1.0;         % Initial values
    dxdt = 0.0;
    t=1.0e-6;
    for i=1:N      % Numerical Scheme
        dxdt = dxdt - (2.0*dxdt/t+t^(lambda-2)*x^n)*dt;
        x=x+dxdt*dt;
        t=i*dt;
        xi(i,j)=t;
        theta(i,j)=x;
        thetap(i,j)=dxdt;
    end
end
```

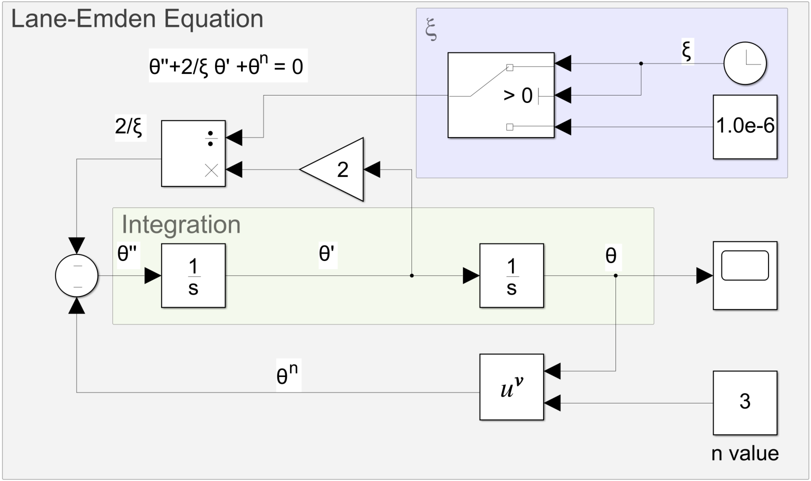


Figure 4: Simulink Model

Nonlinear Dynamics

Mancas and Rosu [8] studied the Emden-Fowler equation,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\eta}{d\xi}\right) = \alpha\xi^\lambda\eta^n,$$

by using a homework problem in Jordan and Smith [6] to represent the equation as a system to two first order equations. Setting

$$X = \frac{\xi\eta'}{\eta} \quad \text{and} \quad Y = \xi^{\lambda-1}\frac{\eta^n}{\eta'}$$

and $\xi = e^t$, one obtains a two-dimensional autonomous system,

$$\begin{aligned} \dot{X} &= -X(1+X-\alpha Y) \\ \dot{Y} &= Y(1+\lambda+nX-\alpha Y). \end{aligned} \tag{1}$$

Here the dot represents $\frac{d}{dt}$.

Nonlinear Systems

We begin with the $n-$dimensional system

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathrm{R}^n. \tag{2}$$

Here $\boldsymbol{f}: \ \mathrm{R}^n \rightarrow \ \mathrm{R}^n$.

We define the equilibrium solutions (fixed points), $\boldsymbol{x}^*$, satisfying $\boldsymbol{f}(\boldsymbol{x}^*) = 0$.

Stability

We are interested in what happens to solutions of the system with initial conditions starting near a fixed point. Let $\boldsymbol{x} = \boldsymbol{x}^* + \boldsymbol{\xi}$ and that initially, $|\boldsymbol{\xi}| \ll 1$.

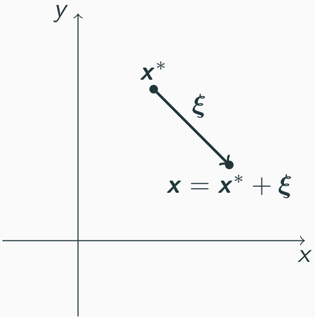


Figure 5: A general point in the plane, which is near the fixed point, in the form $\boldsymbol{x} = \boldsymbol{x}^* + \boldsymbol{\xi}$,

The Evolution of ξ

The change of $\boldsymbol{\xi}$ in time:

$$\mathbf{x}' = \boldsymbol{\xi}'.$$

Next, we have that

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}^* + \boldsymbol{\xi}).$$

Thus, we have that

$$\mathbf{f}(\mathbf{x}^* + \boldsymbol{\xi}) = \mathbf{f}(\mathbf{x}^*) + D\mathbf{f}(\mathbf{x}^*)\boldsymbol{\xi} + O(|\boldsymbol{\xi}|^2).$$

Here $D\mathbf{f}(\mathbf{x})$ is the *Jacobian matrix*, defined as

$$D\mathbf{f}(\mathbf{x}^*) \equiv \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

Summary

Noting that $\boldsymbol{f}(\boldsymbol{x}^*) = 0$, we then have that system (2) becomes

$$\boldsymbol{\xi}' \approx D\boldsymbol{f}(\boldsymbol{x}^*)\boldsymbol{\xi}. \tag{3}$$

The stability of the equilibrium point of the nonlinear system is now reduced to analyzing the behavior of the linearized system given by Equation (3). We investigate the eigenvalues of the Jacobian matrix evaluated at each equilibrium point.

We will demonstrate this procedure with several examples.

Example i

Determine the equilibrium points and their stability for the system

$$\begin{aligned} x' &= -2x - 3xy, \\ y' &= 3y - y^2. \end{aligned} \tag{4}$$

We first determine the fixed points.

$$\begin{aligned} -x(2+3y) &= 0, \\ y(3-y) &= 0. \end{aligned} \tag{5}$$

We see that either $y = 0$ or $y = 3$ and $x = 0$ in either case. So, there are two fixed points: $(0, 0)$ and $(0, 3)$.

The Jacobian matrix is given by

$$Df(x, y) = \begin{pmatrix} -2 - 3y & -3x \\ 0 & 3 - 2y \end{pmatrix}. \tag{6}$$

Example ii

1. Case I Equilibrium point $(0,0)$.

$$D\mathrm{f}(0,0) = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}. \tag{7}$$

Therefore, the linearized equation becomes

$$\boldsymbol{\xi}' = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix} \boldsymbol{\xi}. \tag{8}$$

This is equivalently written out as the system

$$\begin{aligned} \xi_1' &= -2\xi_1, \\ \xi_2' &= 3\xi_2. \end{aligned} \tag{9}$$

This is the linearized system about the origin. The eigenvalues are $\lambda = -2, 3$. So, the origin is a **saddle point**.

Example iii

2. Case II Equilibrium point $(0, 3)$. The Jacobian matrix for this case becomes

$$Df(0,3) = \begin{pmatrix} -11 & 0 \\ 0 & -3 \end{pmatrix}. \tag{10}$$

This is equivalently written out as the system

$$\begin{aligned} \xi_1' &= -11\xi_1, \\ \xi_2' &= -3\xi_2. \end{aligned} \tag{11}$$

The eigenvalues are $\lambda = -11, -3$. So, this fixed point is a **stable node**.

Example iv

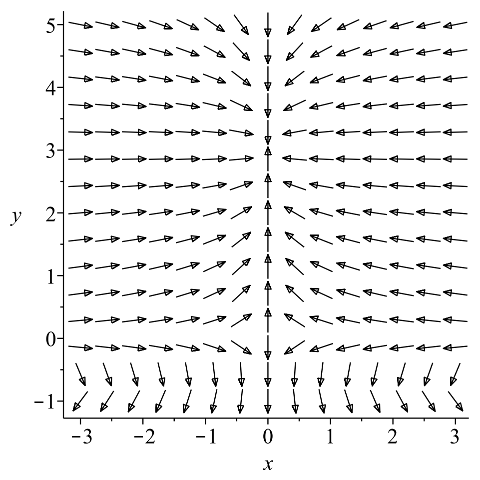


Figure 6: Phase plane for the system $x' = -2x - 3xy$, $y' = 3y - y^2$.

Return to the Emden-Fowler System

$$\begin{aligned} \dot{X} &= -X(1 + X - \alpha Y) \\ \dot{Y} &= Y(1 + \lambda + nX - \alpha Y). \end{aligned} \tag{12}$$

There are four equilibrium solutions,

$$\begin{aligned} (X_0, Y_0) &= (0, 0) \\ (X_0, Y_0) &= (-1, 0) \\ (X_0, Y_0) &= \left(0, \frac{1+\lambda}{\alpha}\right) \\ (X_0, Y_0) &= \left(\frac{\lambda}{1-n}, \frac{\lambda - n + 1}{\alpha(1-n)}\right). \end{aligned} \tag{13}$$

Linear Stability Analysis

Using a linear stability analysis, one needs the Jacobian matrix,

$$J(X,Y) = \begin{pmatrix} -1-2X+\alpha Y & \alpha X \\ nY & 1+\lambda+nX-2\alpha Y \end{pmatrix}.$$

The eigenvalues, μ, satisfy the equation

$$\mu^2 - (\text{tr } J)\mu + \det J = 0. \tag{14}$$

Then, the eigenvalues are given by

$$\mu = \frac{\text{tr}J \pm \sqrt{\text{tr}^2 J - 4\det J}}{2}. \tag{15}$$

Below we will refer to the discriminant, $\Delta = \text{tr}^2 J - 4\det J$.

The Behavior of the Equilibrium Points

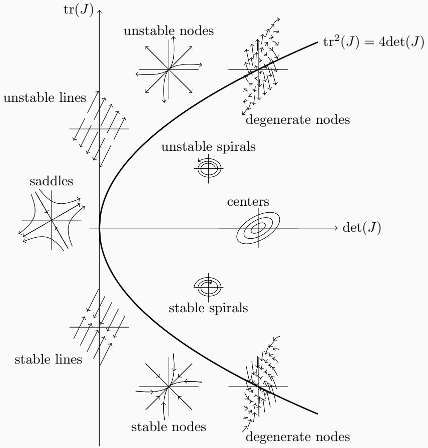
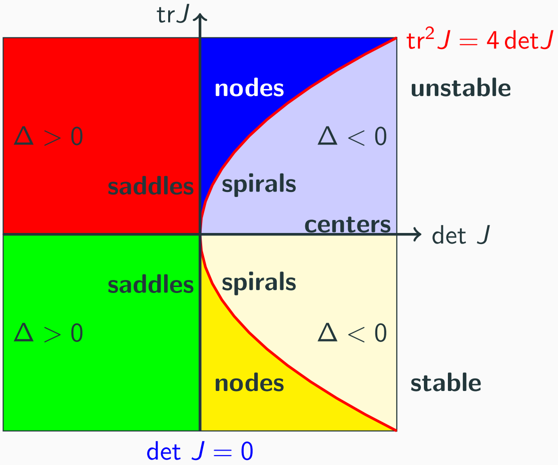


Figure 7: The behavior of equilibrium points in the det − tr plane. The curve $\mathrm{tr}^2(J) = 4\det(J)$ indicates where the discriminant vanishes.

The stability of the equilibrium points depends on the eigenvalues, which in turn depend on the trace and determinant of J.



Frame Title

(X_0, Y_0)	det J	tr J	Δ
$(0,0)$	$-1-\lambda$	λ	$(\lambda+2)^2$
$(-1,0)$	$\lambda - n + 1$	$\lambda - n + 2$	$(\lambda - n)^2$
$(0, \frac{1+\lambda}{\alpha})$	$-\lambda(1+\lambda)$	-1	$(1+2\lambda)^2$
$\left(\frac{\lambda}{1-n}, \frac{\lambda-n+1}{\alpha(1-n)}\right)$	$\frac{\lambda(n-\lambda-1)}{n-1}$	$\frac{2\lambda-n+1}{n-1}$	$\frac{4\lambda n(\lambda-n+1)+(-1+n)^2}{(n-1)^2}$

Table 1: The trace, determinant, and discriminant of the Jacobian matrix for the equilibrium points.

Stability for $X = 0, Y = 0$

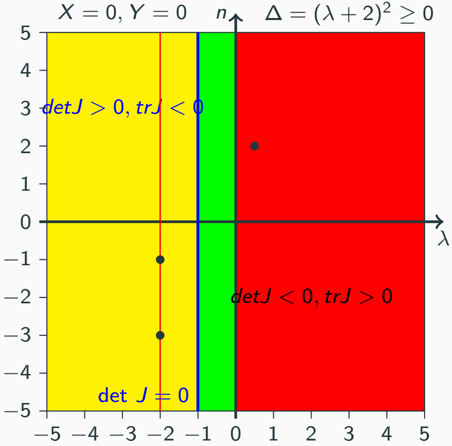


Figure 8: Blue line: $\det\ J = 0$. Red line: $\Delta = 0$. The equilibrium point is a saddle or a stable node. The black dots are examples in [8] for $\alpha = -1$, and $(n, \lambda) = (-2, -3)$, $(n, \lambda) = (-2, -1)$, and $(n, \lambda) = (\frac{1}{2}, 2)$.

Stability for $X = -1, Y = 0$

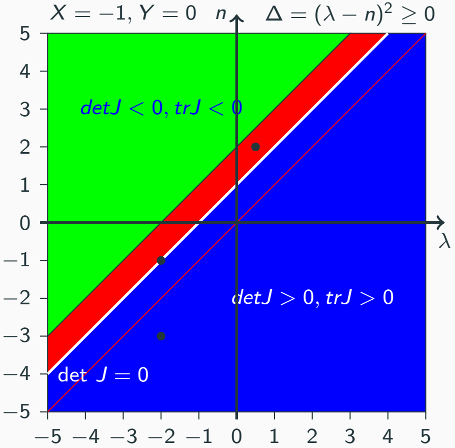


Figure 9: Stability for the equilibrium point $(X, Y) = (-1, 0)$. The white line is where $\det J = 0$. The red line is for $\Delta = 0$. The diagram indicates the equilibrium point can only be a saddle or an unstable node.

Stability for $(X, Y) = \left(\frac{\lambda}{1-n}, \frac{\lambda-n+1}{\alpha(1-n)}\right)$.

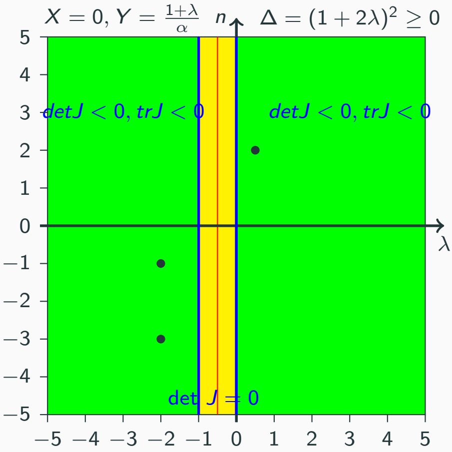


Figure 10: There are two blue lines showing where $\det J = 0$. The red line is for $\Delta = 0$. The diagram indicates the equilibrium point can only be a saddle or a stable node.

Stability for $(X, Y) = \left(\frac{\lambda}{1-n}, \frac{\lambda-n+1}{\alpha(1-n)}\right)$.

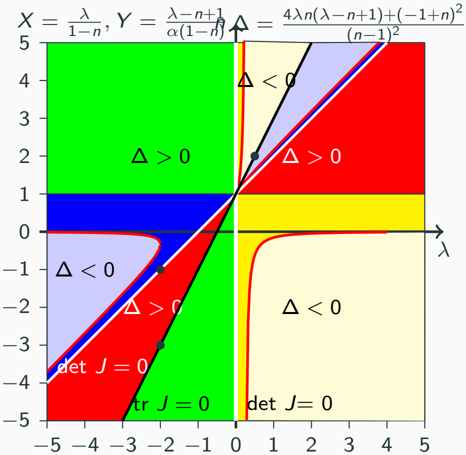


Figure 11: White lines: det $J = 0$. Red curves: $\Delta = 0$. This equilibrium point can be a stable spiral (light yellow), unstable spiral (light blue), or a center (along the black line: tr $J = 0$).

Summary of Analysis

Once we know the behaviors of each equilibrium point near the point, we need to see how the local behaviors connect to give the global behavior.

This can only be done numerically. We can piece together the information from Figures 8-11 to decide what might be interesting. Mancas and Rosu [8] and others were concerned with things like the existence of periodic solutions or if one could construct analytic solutions to the given equation.

In Figure 12 we map out the regions of interest based on the earlier figures.

Regions of Interest

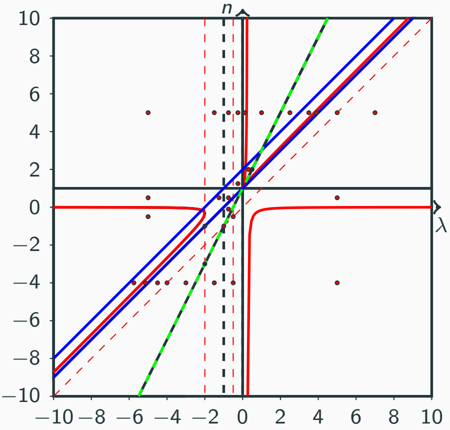
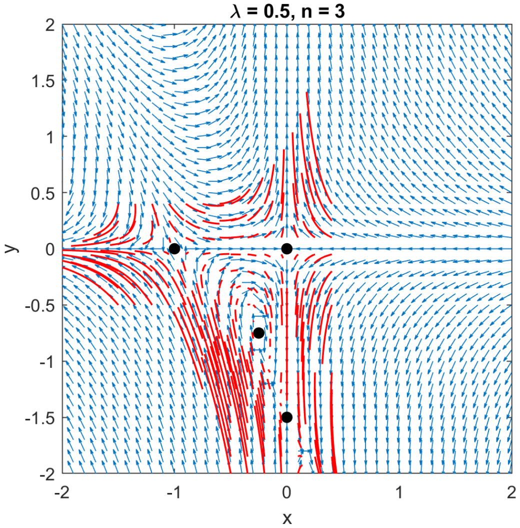
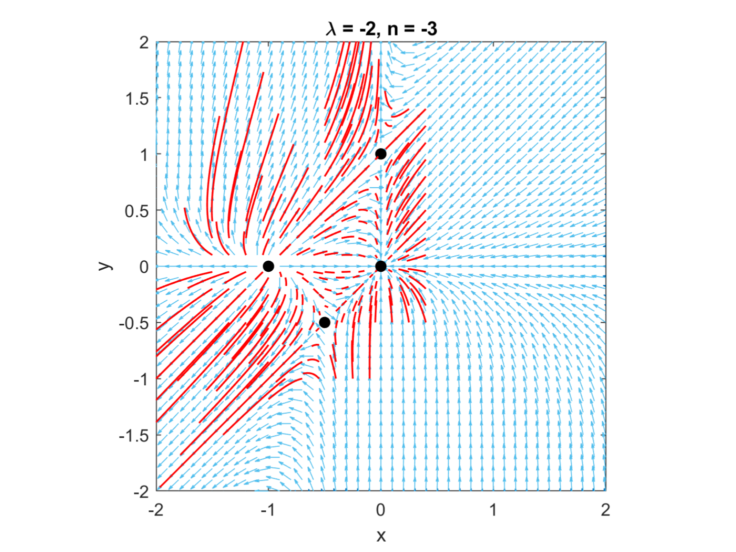


Figure 12: There are about two dozen different regions The red lines are where one of the equilibrium points gives det $J = 0$. The dashed red lines come from the first three equilibrium points and the solid red line comes from the fourth point.

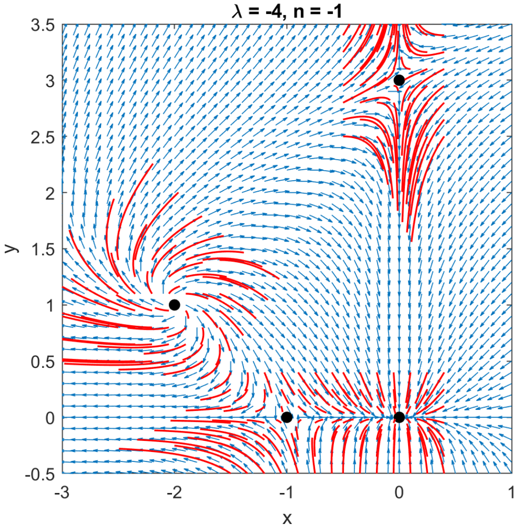
Phase Portrait for $n = 3$, $\lambda = 0.5$



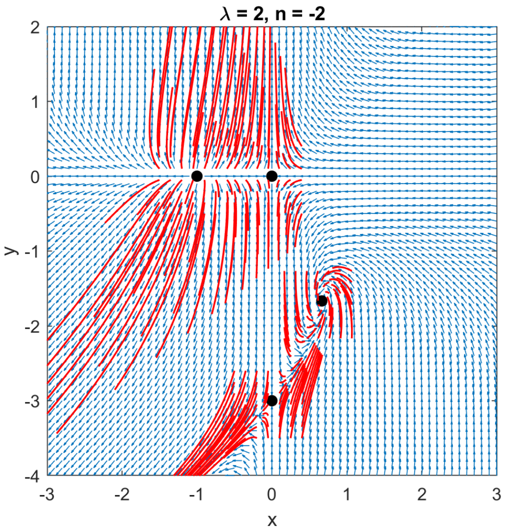
Phase Portrait for $n = -3$, $\lambda = -2$



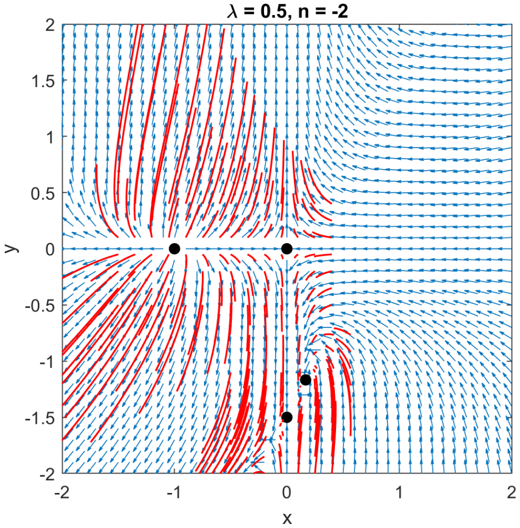
Phase Portrait for $n = -1$, $\lambda = -4$



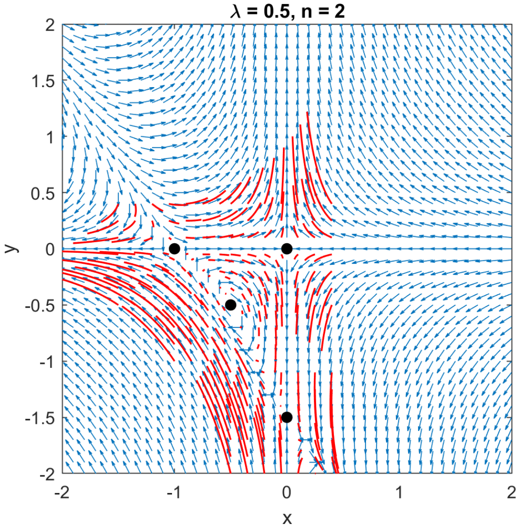
Phase Portrait for $n = -2$, $\lambda = 2$

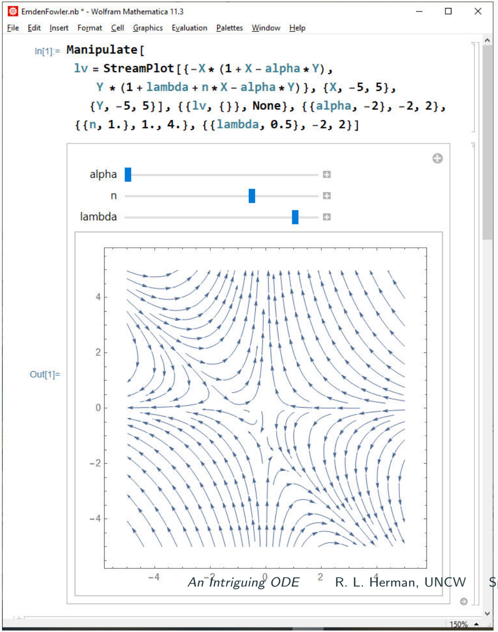


Phase Portrait for $n = -2$, $\lambda = 0.5$



Phase Portrait for $n = 2$, $\lambda = 0.5$




EmdenFowler.nb * - Wolfram Mathematica 11.3
File Edit Insert Format Cell Graphics Evaluation Palettes Window Help
In[1]:= Manipulate[
lv = StreamPlot[{-X * (1 + X - alpha * Y),
Y * (1 + lambda + n * X - alpha * Y)}, {X, -5, 5},
{Y, -5, 5}], {{lv, {}}, None}, {{alpha, -2}, -2, 2},
{{n, 1.}, 1., 4.}, {{lambda, 0.5}, -2, 2}]
alpha
n
lambda
Out[1]=
4
2
0
−2
−4
−4
−2
0
2
4
150%

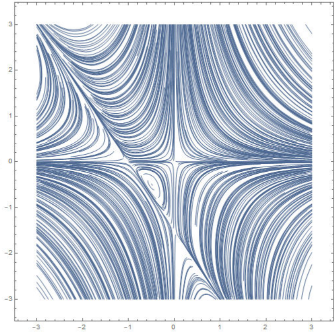
Mathematica - Detailed Plot $n = 2,\ \lambda = 0.5$

EmdenFowler.nb * - Wolfram Mathematica 11.3

File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

In[∘]:=
```
t =
  Map[StreamPlot[{-X * (1 + X + Y), Y * (1 + .5 + 2 * X + Y)},
     {X, -3, 3}, {Y, -3, 3}, StreamStyle → "Line",
     StreamPoints → RandomReal[{-3, 3}, {#, 2}],
     ImageSize → 500] &, ConstantArray[50, 20]];
Show[t]
```

Out[∘]=



Other Approaches - Perturbation Theory

Bender, et al., 1989 set $n = 1 + \delta$, $\delta \ll 1$.

$$y''(x) + \frac{2}{x} y'(x) + [y(x)]^{1+\delta} = 0.$$

Let

$$y(x) = y_0(x) + \delta y_1(x) + \delta^2 y_2(x) + \cdots .$$

Then,

$$\begin{aligned} y_0''(x) + \frac{2}{x} y_0'(x) + y_0(x) &= 0 \\ y_1''(x) + \frac{2}{x} y_1'(x) + y_1(x) &= -y_0(x) \ln y_0(x) \end{aligned}$$

with

$$y_0(0) = 1, y_0'(0) = 0, y_1(0) = 0, y_1'(0) = 0, \ldots .$$

Solutions were found and zeros approximated.

Other Approaches - Lie Symmetry Analysis

Symmetry analysis is applied to generalizations,

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + f(y) = 0.$$

- $f(y) = (y^2 - C)^{\frac{3}{2}}$, Chandrasekhar - white dwarfs
- $f(y) = e^y$, - isothermal gas spheres.

One views the differential equation as a general surface in the space (x, y, y', y'') and seeks symmetry groups via Lie point generators of the symmetry group. The generators are then used to obtain solutions.

Summary

- There are interesting problems.
- No telling what you'll learn from the sources.
- There are multiple methods that can be used.
- Nonlinear dynamics is interesting.
- MATLAB, Simulink, Mathematica, ... $\LaTeX$.
- Numerical solutions, perturbation theory, symmetry analysis.
- It can't hurt to do it yourself.

Thanks for listening!

References i

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