Heaviside's Operational Calculus, Telegraphy, and the Laplace Transform

Graduate Seminar, Fall 2023

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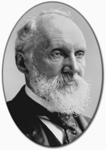
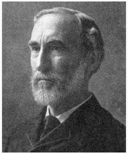
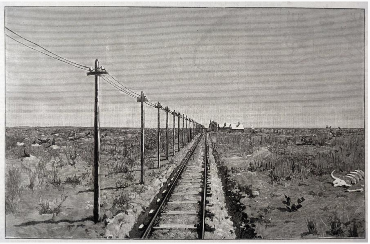
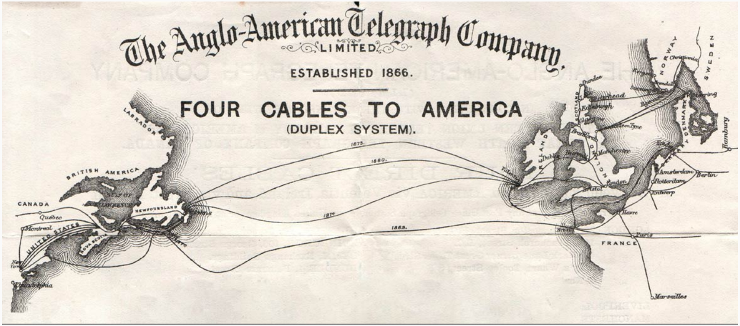
     

Outline





The Anglo-American Telegraph Company,
LIMITED
ESTABLISHED 1866.
FOUR CABLES TO AMERICA
(DUPLEX SYSTEM).
LABRADOR
BRITISH AMERICA
CANADA
NEWFOUNDLAND
UNITED STATES
NOVA SCOTIA
IRELAND
FRANCE
NORWAY
SWEDEN
DENMARK
1873.
1880.
1874
1865.

Rise of the Telegraph

- 1833, Carl Friedrich Gauss (1777-1855) and Eduard Friedrich Weber (1806-1871) recognised that electric signals could be used to pass messages.
- Adapted by Sir William Fothergill Cooke (1806-1879) and Charles Wheatstone (1802-1875),
- The first public electric telegraph was established in 1837 along the Great Western railway from Paddington to West Drayton.
- Adoption of Greenwich Mean Time (GMT).

Figure 1: Wheatstone and Cooke.

Rise of the Telegraph

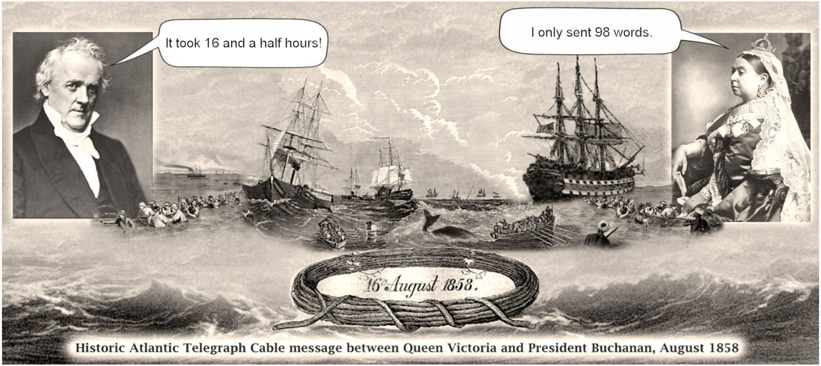
- Transmission wires along railway track supported poles.
- Samuel Morse, Morse code, 1838.
- Dreamed thousands of miles of cable.
- Morse insulated wire with tarred hemp, New York Harbour, 1842,
- He telegraphed through submerged wire.
- 1st commercial line, 1844 in U.S.
- 1850 Great Britain to France.

Transantlantic Telegraph

- America-Nova Scotia-Newfoundland - Largest , Frederick W. Gisbone. - 1853, not profitable.
- 1857 Cyrus West Field - New York, Newfoundland & London Telegraph Co.
- 1857/8 Several attempts.
- Whitehouse vs Thomson.
- HMS Agamemnon and USS Niagara
- Aug, 5 - Iceland, cable broke twice.
- 1858 Ships headed towards each other.
- Cable broke - 6 km, 100 km, 370 km.
- Jul 29, 1858, Got to ports Aug 4/5: Agamemnon to Valentia, Ireland and Niagara to Trinity Bay.

The First Transatlantic Message

- Aug 16, 1858 - Queen Victoria and President Buchanan exchanged messages.
- Two char/min - 1st message, 16 hrs.
- There has to be a better way! - Eventually, July 1866.



- Whitehouse cranked voltage from 600V to 2000V, frying the insulation. Dismissed Aug. 17.

William Thomson (1824-1907)

- Father, James Thomson, taught math in Belfast and Univ. of Glasgow.
- William attended Univ. of Glasgow, 1834 (at 10).
- Read Jean-Baptiste-Joseph Fourier.
- First two articles, at 16-17, defended Fourier.
- Cambridge, 1841-5, earned B.A. with high honours.
- In 1845, obtained George Green's essay and went to Paris next day.
- Chair of natural philosophy (physics) at the Univ. of Glasgow at 22.

WILLIAM THOMSON: THE YOUNG PROFESSOR

Thomson became interested in the telegraphy problem in 1854: Exchanged letters with George Gabriel Stokes (1819-1903) (Thomson 1856).

Thomson's 1855 Paper

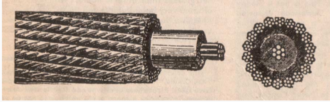
III. "On the Theory of the Electric Telegraph." By Professor William Thomson, F.R.S. Received May 3, 1855.

The following investigation was commenced in consequence of a letter received by the author from Prof. Stokes, dated Oct. 16, 1854. It is now communicated to the Royal Society, although only in an incomplete form, as it may serve to indicate some important practical applications of the theory, especially in estimating the dimensions of telegraph wires and cables required for long distances; and the author reserves a more complete development and illustration of the mathematical parts of the investigation for a paper on the conduction of Electricity and Heat through solids, which he intends to lay before the Royal Society on another occasion.

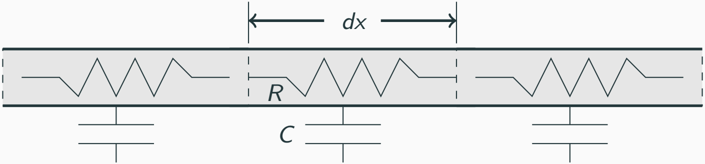
Extract from a letter to Prof. Stokes, *dated Largs, Oct.* 28, 1854.

"Let c be the electro-statical capacity per unit of length of the wire; that is, let c be such that clv is the quantity of electricity

William Thomson's Telegraph Theory - 1855



Treat the coaxial cable as a long, thin conductor, perfectly electrically insulated.



Think of the cable as a network of resistances and electrical capacity (capacitance) and use Kirchoff's laws on an infinitesimal section to derive an equation for the voltage, $v(x, t)$. [Ohm - 1827, Kirchoff - 1845.]

Thomson's Diffusion Equation

This resulted in a diffusion equation:

$$\frac{\partial v^2}{\partial x^2} = RC\frac{\partial v}{\partial t}. \tag{1}$$

It was Fourier's heat equation with solution (Thomson 1856),

$$v = \frac{Q\sqrt{R}}{\sqrt{\pi Ct}} e^{-RCx^2/4t}.$$

The maximum effect is at position x and time $t = \frac{1}{6}RCx^2$. This is **Thomson's law of squares**. Examples in Rayleigh's *Theory of Sound.*

Thomson solved several special cases in his correspondence with Stokes as recalled by Thomson (1856). Thomson's theory had many practical applications.

Further Develpoments

- Thomson had the first teaching laboratory,
- Engaged his students in testing materials and his ideas.
- Used the theory/experiment to understand underwater telegraphy.
- Explained the speed of the current in a telegraph cable,
- Dispersion caused signals of low frequency to diffuse less.

Stokes solved the more general case

$$\begin{aligned} v(x,0) &= 0, \quad 0 < x < \infty \\ v(0,t) &= f(t), \quad 0 < t < \infty, \end{aligned}$$

arriving at the solution (Nahin 2002)

$$v(x,t) = \frac{x}{2\sqrt{\pi}} \int_0^t (t - t')^{-\frac{3}{2}} e^{-x^2/4(t-t')} f(t')\, dt'.$$

William Thomson - a.k.a Lord Kelvin

This was in the backdrop of the Atlantic Cable Project (Nahin 2002).

- Developed the theory, designed experiments, and obtained patents.
- Was instrumental to the success of the trans-Atlantic cable, completed 1866, after disputes with Whitehouse. (Crossland 2008; Flood, McCartney, and Whitaker 2008; Bart and Bart 2008).
- For his work on the trans-Atlantic telegraph project:
    - Knighted by Queen Victoria, becoming Sir William Thomson, 1866.
    - Recognized for achievements in thermodynamics becoming Baron Kelvin, of Largs, 1892. (Crossland 2008)

Thomson's theory of the electric telegraph remained the main theory for decades. It worked fine for long underwater cables, but to transmit human conversation, the diffusion was far too much.

Maxwell's Theory of Electricity and Magnetism

During this time scientists were beginning to move from the mechanical world of Newton and Lagrange to the world of Faraday, Oersted, Ampere, and others.

- James Clerk Maxwell (1831-1879)
- Michael Faraday (1791-1867) encouraged Maxwell.
- "A Dynamical Theory of the Electromagnetic Field," EM waves, Maxwell (1865).
- "A Treatise on Electricity and Magnetism," 1873.
- Promoters of Maxwell's work: G. F. Fitzgerald (1851-1901), O. Heaviside (1850-1925), and O. Lodge (1851-1940). The Maxwellians (O'Hara and Pricha 1987; Hunt 1991).
- The race was on to produce electromagnetic waves, Hertz (1857-1894).
- Maxwell's theory reworked by Heaviside.

Challenge to Thomson's Theory

- The story of the attempts to connect continents with telegraph cables and Thomson's role is described by Hunt (2018, 2021, 2012).
- The subsequent contributions of Heaviside acan be found in (Nahin 2002).
- In 1876 Heaviside derived the telegrapher's equation independently and updated Thomson's diffusion theory by insisting that self-inductance was important.
- This was contrary to what people working on underwater telegraphy believed.
- It led to a few disputes.

Figure 2: Who was Oliver Heaviside?

Oliver Heaviside (1850-1925)

- Heaviside left school at sixteen. (Nahin 2002)
- He studied at home for two years.
- Worked as telegraph operator, Danish-Norwegian-English Telegraph Co., advice from uncle C. Wheatstone, 1868.
- He was transferred to Newcastle-on-Tyne, 1870, and later appointed Chief Operator.
- He left in 1874. Only job he would ever have.
- He spent the next couple of years working on electric theory.
- He studied and reformulated Maxwell's theory.

Note: Heaviside and Josiah Gibbs gave us Vector Analysis and opposed quaternions introduced by Hamilton and promoted by Tait. He gave us Maxwell's Equations.

Oliver Heaviside (1850-1925)

- Heaviside began publishing in 1872.
- He furthered Thomson's theory, 1876.
- Derived the telegraph equation.
- Self-induction is important in telegraphy.
- Others opposed him on this.
- He was asked to stop publishing for *The Electrician* in 1887. (Watson-Watt 1950; Edge 1983; Nahin 1991, 2002; Giorello and Sinigaglia 2005; Hunt 2012; Mahon 2017).
- Heaviside did have some supporters including Thomson and Maxwell.

Heaviside’s Theory

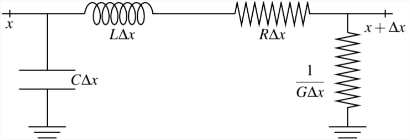


Figure 3: A section of transmission Line to find the potential drop from x to $x + \Delta x$ across an inductor and resistor with leakage.

Using Kirchoff’s voltage and current laws, consider the voltage drops from x to $x + \Delta x$ across the resistor and the inductor:

$$\Delta v = -iR\,\Delta x - \frac{\partial i}{\partial t} L\,\Delta x.$$

The current can leak out. The overall change in current is given by

$$\Delta i = -vG\,\Delta x - \frac{\partial v}{\partial t} C\,\Delta x.$$

The Telegrapher's Equation

Dividing by Δx and letting Δx approach zero, we have the two equations

$$\begin{aligned} v_x + iR + Li_t &= 0 \\ Cv_t + Gv + i_x &= 0. \end{aligned} \tag{2}$$

Differentiating the first equation with respect to x and using i_x from the second equation, leads to an equation for the voltage,

$$\frac{1}{LC}\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial^2 t} + \left(\frac{R}{L} + \frac{G}{C}\right)\frac{\partial v}{\partial t} + \frac{GR}{LC}v. \tag{3}$$

A similar equation can be derived for the current.[1]

1. Gray (1923) wrote *The Equation of Telegraphy* comparing known solutions of ,

$$\frac{\partial^2 V}{\partial t^2} + 2\gamma\frac{\partial V}{\partial t} = a^2\frac{\partial^2 V}{\partial x^2},$$

and used the Riemann-Green method.

Heaviside's Operational Calculus

- Used to solve partial differential equations.
- Methods were criticized - not being rigorous and hard to understand.
- First people to publish justifications of Heaviside's methods: Bromwich (1917) and Wagner (1916). (Lützen 1979)
- Both used complex integrals.
- Bromwich - applications from the *Theory of Sound* (Rayleigh 1894) and equation similar to telegrapher's equation using a Green's function.
- Attempted to justify Heaviside's work and eventually the Laplace transform emerged. (Lützen 1979, 2012).

Operational methods for differential equations and the exploration of fractional differentiation had been studied for a number of years going back to the work of Euler and Leibniz (Deakin 1981; Petrova 1987).

Some of this is summarized in Moore's 1921 text (Moore 1971) and from Carslaw and Jaeger's 1941 book on operational methods (Bateman 1942; Carslaw and Jaeger 1941).

Example: Edmund T. Whittaker Obituary for Heaviside

Edmund T. Whittaker (1873-1956) describes how Heaviside would use operational calculus[2] to solve the differential equation (Moore 1971)

$$\frac{d^2y(t)}{dt^2} + k^2y(t) = 0. \tag{4}$$

Let $D = \frac{d}{dt}$. We write symbolically,

$$(D^2 + k^2)y(t) = 0.$$

Now, manipulate algebraically: Multiply by D^{-2},

$$(1 + k^2D^{-2})y(t) = D^{-2}(0).$$

What is $D^{-2}(0)$?

2. According to Whittaker, Heaviside was accustomed to using symbolic differential operators. Boole (1872) devoted two chapters in *A Treatise on Differential Equations* to symbolic methods.

Example (cont'd)

$D^{-2}(0)$ is the most general function whose second derivative vanishes.

It is $a + bt$, where a and b are arbitrary constants. So,

$$(1 + k^2 D^{-2}) y(t) = D^{-2}(0) = a + bt.$$

Solve for $y(t)$, using a series expansion,

$$\begin{aligned} y(t) &= (1 + k^2 D^{-2})^{-1}(a + bt) \\ &= (1 - k^2 D^{-2} + k^4 D^{-4} - \cdots)(a + bt). \qquad (5) \end{aligned}$$

We apply D^{-n} to the functions 1 and t, where

$$D^{-1} = \int_0^t d\tau, \qquad D^{-2} = \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2, \ldots.$$

So, we have $D^{-1} t = \frac{1}{2} t^2$, etc.

Example (cont'd)

Formally, using series expansions, we have

$$\begin{aligned} y(t) &= (1+k^2D^{-2})^{-1}(a+bt) \\ &= (1-k^2D^{-2}+k^4D^{-4}-\cdots)(a+bt) \\ &= a(1-k^2D^{-2}+k^4D^{-4}-\cdots)(1) \\ &\quad +b(1-k^2D^{-2}+k^4D^{-4}-\cdots)(t). \\ &= a\left(1-\frac{k^2t^2}{2!}+\frac{k^4t^4}{4!}-\cdots\right)+b\left(t-\frac{k^2t^3}{3!}+\frac{k^4t^5}{5!}-\cdots\right) \\ &= a\cos kt+\frac{b}{k}\sin kt. \qquad (6) \end{aligned}$$

Since a and b are arbitrary constants, we have found the general solution to the differential equation $y''+k^2y=0$.

This method works for many linear differential equations, even partial differential equations like the heat and telegrapher's equations.

Heat Equation Example

Heaviside solved Thomson's equation for cables,

$$\frac{\partial^2 V}{\partial x^2} = k\frac{\partial V}{\partial t}, \tag{7}$$

where $k = RC$.

Heaviside let $p = \frac{\partial}{\partial t}$, giving

$$\frac{\partial^2 V}{\partial x^2} = pkV. \tag{8}$$

Treating p as algebraic, solve the second order differential equation:

$$V = e^{-(pk)^{1/2}x}V_0,$$

assuming bounded solutions.

What does $e^{-\sqrt{pk}} = e^{-\sqrt{k\frac{\partial}{\partial t}}}$ mean?

Heat Equation Example (cont'd)

The solution can be used to find the surface gradient at $x = 0$.

First, we note that if $\frac{\partial^2}{\partial x^2} = pk$, then $\frac{\partial}{\partial x} = (pk)^{1/2}$. Then,

$$\begin{aligned} \left(\frac{\partial V}{\partial x}\right)_{x=0} &= \left(-(pk)^{1/2} e^{-(pk)^{1/2}x} V_0\right)_{x=0} \\ &= -\sqrt{pk}\, V_0 = -k^{1/2} \left(\frac{\partial}{\partial t}\right)^{1/2} V_0. \qquad (9) \end{aligned}$$

Heaviside often obtained expressions involving fractional derivatives.

In fact, many before Heaviside spent time trying to make sense out of nonstandard derivatives and integrals. Ross (1977) describes some of the early work on fractional derivatives.

Fractional differentation

- In 1695 Leibniz communicated about fractional derivatives to Johann Bernoulli and l'Hôpital.
- In 1729 Euler communicated to Goldbach the general form

$$\frac{d^n x^p}{dx^n} = \frac{\Gamma(p+1)}{\Gamma(p-n+1)} x^{p-n}, \tag{10}$$

  using the Gamma function, $\Gamma(n) = n!$ for integers n.
- One definition (Riemann-Liouville)

$$f^{(q)}(x) = \frac{1}{\Gamma(k-q)} \frac{d^k}{dx^k} \int_a^x (x-t)^{k-q-1} f(t)\, dt.$$

- From Euler's formula (10) we have for $n = \frac{1}{2}$

$$D^{1/2} \cdot 1 = \frac{\Gamma(1)}{\Gamma(\frac{1}{2})} t^{-1/2} = \frac{1}{\sqrt{\pi t}}.$$

Heat Equation Example 2

Heaviside applied operational methods to the heat equation for more complicated problems.

Consider the heat equation

$$\frac{\partial^2 V}{\partial x^2} = k\frac{\partial V}{\partial t}, \tag{11}$$

with $V = 0$, $t < 0$, and boundary condition

$$V_0 - V = h\frac{\partial V}{\partial x}, \quad x = 0.$$

Find the solution such that $V = V_1$ at $x = 0$.

Defining $D = \frac{d}{dx}$, the operational form of the heat equation is

$$D^2 V = kpV. \tag{12}$$

Heat Equation Example 2 (cont'd)

The boundary condition can be written at $x = 0$, using $DV = \sqrt{kp}V$, as

$$V_0 - V_1 = h\frac{\partial V}{\partial x} = h\sqrt{kp}V_1.$$

Solving for V_1 and defining $a = kh^2$, we have

$$V_1 = \frac{1}{1+\sqrt{ap}}V_0.$$

So, how do we work with this solution? We symbolically expand $(1+\sqrt{ap})^{-1}$ as a geometric series, ignoring convergence issues.

This is one place where Heaviside managed to upset mathematicians. Heaviside saw mathematics as an experimental science. Edge (1983) quotes him, "Mathematics is of two kinds, Rigorous and Physical. The former is Narrow: the latter Bold and Broad. To have to stop to formulate rigorous demonstrations would put a stop to most physico-mathematical enquiries. Am I to refuse to eat because I do not fully understand the mechanism of digestion?"

Heat Equation Example 2 (cont'd)

There are two ways to do the expansion. First, we have

$$\begin{aligned} V_1 &= \frac{1}{1+\sqrt{ap}} V_0 \\ &= \left[1 - \sqrt{ap} + ap - (ap)^{3/2} + \cdots\right] V_0 \\ &= \left[1 - \sqrt{ap} - (ap)^{3/2} - (ap)^{5/2} - \cdots\right] V_0 \\ &= \left[1 - \sum_{k=0}^{\infty} (ap)^{k+\frac{1}{2}}\right] V_0. \end{aligned} \tag{13}$$

Note that the positive terms in the expansion simply using

$$p^n V_0 = \begin{cases} V_0, & n = 0, \\ 0, & n = 1, 2, \ldots \end{cases}$$

since derivatives of a constant vanish.

Heat Equation Example 2 (cont'd)

The second way to perform the expansion is in powers of $1/\sqrt{ap}$:

$$\begin{aligned} V_1 &= \frac{1}{1+\sqrt{ap}} V_0 \\ &= \frac{1}{\sqrt{ap}} \left[\frac{1}{1+\frac{1}{\sqrt{ap}}} \right] V_0 \\ &= \frac{1}{\sqrt{ap}} \left[1 - (ap)^{-1/2} + (ap)^{-1} - (ap)^{-3/2} + \cdots \right] V_0 \\ &= \left[(ap)^{-1/2} - (ap)^{-1} + (ap)^{-3/2} - \cdots \right] V_0 \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} (ap)^{-n/2} V_0. \end{aligned} \tag{14}$$

For both series we need to perform fractional differentiation.

Heat Equation Example 2 (cont'd)

Using Euler's derivative formula

$$\frac{d^n x^p}{dx^n} = \frac{\Gamma(p+1)}{\Gamma(p-n+1)} x^{p-n}, \tag{15}$$

we have for the first series (13),

$$p^{k+\frac{1}{2}} \cdot 1 = \frac{t^{-\frac{1}{2}-k}}{\Gamma\left(\frac{1}{2}-k\right)}, \quad k = 0, 1, 2, \ldots.$$

For the second series (14), we need

$$p^{-n/2} \cdot 1 = \frac{1}{\Gamma(\frac{n}{2}+1)} t^{n/2}, \quad n = 1, 2, \ldots.$$

These results can be written out in an explicit form using properties of Gamma functions.

Heat Equation Example 2 (cont'd)

We can show

$$\frac{1}{\Gamma\left(\frac{1}{2}-k\right)} = \frac{(-1)^k}{\pi}\Gamma\left(k+\frac{1}{2}\right) = \frac{(-1)^k}{\sqrt{\pi}}\frac{(2k)!}{2^{2k}k!}. \tag{16}$$

So, the solution in the first case is

$$\begin{aligned} V_1 &= \left[1-\sum_{k=0}^{\infty}(ap)^{k+\frac{1}{2}}\right]V_0 & (17)\\ &= V_0\left[1-\left(\frac{a}{\pi t}\right)^{1/2}\sum_{k=0}^{\infty}(-1)^k\frac{(2k)!}{2^{2k}k!}\left(\frac{a}{t}\right)^k\right] \\ &= V_0\left[1-\left(\frac{a}{\pi t}\right)^{1/2}\left(1-\frac{a}{2t}+1\cdot 3\left(\frac{a}{2t}\right)^2-\cdots\right)\right]. & (18)\end{aligned}$$

This is an asymptotic series for large t.

Heat Equation Example 2 (cont'd)

For the second series (14), we can show for small t

$$\begin{aligned} V_1 &= \sum_{n=1}^{\infty}(-1)^{n+1}(ap)^{-n/2}V_0 \\ &= 2V_0\sqrt{\frac{t}{\pi a}}\sum_{k=0}^{\infty}\frac{2^k}{(2k+1)!!}\left(\frac{t}{a}\right)^k - V_0\sum_{k=1}^{\infty}\frac{(t/a)^k}{k!} \\ &= 2V_0\sqrt{\frac{t}{\pi a}}\left[1+\frac{2}{3}\,\frac{t}{a}+\frac{4\,t^2}{15\,a^2}+\frac{8\,t^3}{105\,a^3}+\frac{16\,t^4}{945\,a^4}+\cdots\right]+V_0\left[1-e^{t/a}\right]. \end{aligned} \tag{19}$$

Summing the infinite series, we have the solution

$$\begin{aligned} V_1 &= V_0\left[e^{\frac{t}{a}}\left(1-\operatorname{erfc}\left(\sqrt{\frac{t}{a}}\right)\right)+1-e^{t/a}\right] \\ &= V_0\left[1-e^{\frac{t}{a}}\operatorname{erfc}\left(\sqrt{\frac{t}{a}}\right)\right]. \end{aligned} \tag{20}$$

Watson's Lemma

The large t result can be understood using (Bender and Orszag 1999).

Watson's Lemma - Asymptotic Expansions Let $f(t)$ be continuous on the interval $0 \leq t \leq b$ and have the asymptotic expansion[3]

$$f(t) \sim t^{\alpha}(a_0 + a_1 t^{\beta} + a_2 t^{2\beta} + \cdots) = t^{\alpha} \sum_{n=0}^{\infty} a_n t^{n\beta}$$

as $t \to 0^+$ and for $\alpha > -1$, $\beta > 0$. Then,

$$\int_0^b f(t) e^{-xt}\, dt \sim \sum_{n=0}^{\infty} \frac{a_n \Gamma(\alpha + \beta n + 1)}{x^{\alpha+\beta n+1}} \quad \text{as } x \to \infty. \tag{21}$$

3. The power series $\sum_{n=0}^{\infty} a_n (t - t_0)^n$ is asymptotic to $f(t)$ if

$$\left| f(t) - \sum_{n=0}^{N} a_n (t - t_0)^n \right| \ll |t - t_0|^N$$

as $t \to t_0$ for every N.

Application of Watson's Lemma

We apply Watson's Lemma (21) to the complementary error function,

$$\operatorname{erfc}(\lambda) = \frac{2}{\sqrt{\pi}} \int_\lambda^\infty e^{-s^2}\, ds,$$

after the variable substitution, $\tau = 2(s-\lambda)$. Then,

$$\begin{aligned}\operatorname{erfc}(\lambda) &= \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-(\tau/2+\lambda)^2}\, d\tau \\ &= \frac{e^{-\lambda^2}}{\sqrt{\pi}} \int_0^\infty e^{-\tau^2/4} e^{-\lambda\tau}\, d\tau.\end{aligned}$$

Then, we identify

$$f(\tau) = e^{-\tau^2/4} = \sum_{n=0}^\infty \frac{(-1)^n \tau^{2n}}{n!4^n}$$

in Watson's Lemma.

Application of Watson's Lemma (cont'd)

Furthermore, we have $\alpha = 0$, $\beta = 2$, and $a_n = \frac{(-1)^n}{n!4^n}$. Therefore,

$$
\begin{aligned}
\operatorname{erfc}(\lambda) &\sim \frac{e^{-\lambda^2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{a_n \Gamma(\alpha + \beta n + 1)}{\lambda^{\alpha+\beta n+1}} \quad \text{as } \lambda \to \infty \\
&= \frac{e^{-\lambda^2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!4^n} \frac{\Gamma(2n+1)}{\lambda^{2n+1}} \\
&= \frac{e^{-\lambda^2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \lambda^{2n+1}} \frac{(2n)!}{2^n n!} \\
&= \frac{e^{-\lambda^2}}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \lambda^{2n+1}} \frac{(2n+1)!!}{2n+1} \\
&= \frac{e^{-\lambda^2}}{\sqrt{\pi}} \left(\frac{1}{\lambda} - \frac{1}{2\lambda^3} + \frac{3}{4\lambda^5} - \frac{15}{8\lambda^7} + \frac{105}{16\lambda^9} \dots \right). \qquad (22)
\end{aligned}
$$

Heat Equation Example 2 (cont'd)

Letting $\lambda = \sqrt{\frac{t}{a}}$, we have for the Heaviside solution (20)

$$\begin{aligned} V_1 &= V_0\left[1 - e^{\frac{t}{a}}\operatorname{erfc}\left(\sqrt{\frac{t}{a}}\right)\right]. \\ &\sim V_0\left[1 - \frac{1}{\sqrt{\pi}}\left(\left(\frac{a}{t}\right)^{1/2} - \frac{1}{2}\left(\frac{a}{t}\right)^{3/2} + \frac{3}{4}\left(\frac{a}{t}\right)^{5/2} - \dots\right)\right] \\ &= V_0\left[1 - \sqrt{\frac{a}{\pi t}}\left(1 - \frac{1}{2}\left(\frac{a}{t}\right) + \frac{3}{4}\left(\frac{a}{t}\right)^2 - \dots\right)\right]. \end{aligned} \tag{23}$$

This is the large t expansion found earlier in Equation (18).

Therefore, the solution of the second Heaviside example Equation (20),

$$V_1 = V_0\left[1 - e^{\frac{t}{a}}\operatorname{erfc}\left(\sqrt{\frac{t}{a}}\right)\right],$$

agrees with the solution obtained using operational calculus.

Age of the Earth

Thomson was interested in problems about the age of the Earth and Sun.

When he was sixteen he wrote that measuring the rate of heat loss from the surface of the Earth could put a bound on the age of the Earth (England, Molnar, and Richter 2007). This interest might have been sparked by reading Fourier's works.

Some of the first quantitative studies of the heat equation were by Fourier (Fourier 1808, 1820, 1822). Fourier had written on the temperature of the Earth and the diffusion of heat in a spherical solid (Godard 2017).

He later wrote a general paper about terrestrial temperatures (Fourier 1824b), which was reprinted (Fourier 1827) and translated in 1837 (Fourier 1824a). This has led to some misconceptions about his role in the origins of the greenhouse effect (Fleming 1999).

Age of the Earth (cont'd)

Naturally Thomson (1862) would use Fourier's work and in 1862 he predicted the age of the Earth based on the heat equation.

In the mid-1800's estimates of the age of the Earth went from a few thousand years to hundreds of millions based on geological estimates. Also, Darwin's theory of evolution came out in 1859.

Assuming an initial high temperature and constant diffusivity, Thomson asked how long it would take to reach the current temperature gradient at the Earth's surface of 1°F/50 ft. He came up with 98 million years (England, Molnar, and Richter 2007; Nahin 1985; Harrison 1987).

This was not long enough according to the geologists. A debate between physicists and geologists ensued based on Thomson's estimates (Jackson 2008).

Age of the Earth (cont'd)

Thomson's theory was accepted by the physics community for decades until in 1895 John Perry (1850-1920), a former assistant of Thomson, challenged Lord Kelvin (Perry 1895; England, Molnar, and Richter 2007; Shipley 2001).

Perry challenged Kelvin's assumptions: the thermal conductivity may not be constant. He found an increase in the age estimate.

This led to a debate amongst supporters of Thomson vs those of Perry.

Peter Guthrie Tait (1831 - 1901) sided with Kelvin and Heaviside took up the problem using his operational mathematics, deriving both Kelvin's and Perry's estimates.

Age of the Earth (cont'd)

Heaviside even opened the second volume of his *Electromagnetic Theory* (Heaviside 1922) with a chapter on the Age of the Earth.

It is interesting that Heaviside used a similar analysis of the diffusion equation to arrive at the age of the Earth using Thomson's data. Then, he took Perry's idea of a nonconstant diffusivity leading to an equation of the form $V_1 = \frac{1}{1+\sqrt{ap}} V_0$ as described in more detail in (Nahin 1985).

This allowed Perry (1895) to obtain a value for the age of the Earth of more than three times Thomson's estimate of 100 million years (Nahin 1985; Shipley 2001).

The debate continued for many years later (Jeffreys 1916, 1927).

The Evolution of Operational Calculus

Heaviside was not the first to use symbolic methods (Cooper 1952). However, he did propel its use, especially amongst those who choose to leave the rigor to others.

There were several efforts to either explain or bring rigor and prove that there was more to Heaviside's methods that might be more palatable to the mathematicians of the day.

Several definitions of Laplace transforms emerged along with contour integral methods such as the Bromwich integral. Van der Pol and Bremmer (1950) attribute the complex integral to Riemann in 1859. They also uses a two-sided Laplace transform throughout the book.

Early papers on the subject were written by Bromwich (1917), Carson Carson (1922), Van der Pol (1929), and Bateman (Bateman 1904). For example Bateman (1904) refers to Pincherle's book (Pincherle and Amaldi 1901) and his inversion formula.

Origins of Laplace Transforms

There are many papers attempting to trace the origins of the Laplace transform.

Deakin (1981, 1982) wrote two in-depth papers tracing the use of integrals to solve differential equations.

These include the appearance of integrals similar to Laplace and Mellin types in the works of Euler, Lagrange, and Laplace.

The origins of solving differential equations using integrals dates back to to Euler (1707-1783). Euler (1768) considered solutions in the form $y(u) = \int [K(u) + Q(x)]^m P(x)\, dx$ and Euler (1744) used the form $\int e^{ax} X(x)\, dx$.

The Laplace transform is named after Pierre-Simon, Marquis de Laplace (1749-1827) based on his work on probability theory (Laplace 1782).

Laplace Transforms

Eventually, the definition of Laplace transforms became standardized and Tables of Laplace Transforms became common such as the Bateman Project (Erdélyi et al. 1954).

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}\,dt, \quad s > 0, \tag{24}$$

where $\lim_{t\to\infty} f(t)e^{-st} = 0$ to guarantee convergence of the integral.

The inverse Laplace transform is obtained using the Bromwich integral (Herman 2016), or Fourier-Mellin integral,

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\,ds. \tag{25}$$

Laplace Transforms and The Heat Equation

We solve Heaviside's heat equation examples using the Laplace transform.

We seek a solution of the heat equation on a semi-infinite interval with either a fixed or a mixed boundary condition at $x = 0$:

$$\frac{\partial^2 u}{\partial x^2} = k \frac{\partial u}{\partial t}, \quad x > 0, t > 0, \quad u(x, 0) = 0, \tag{26}$$

$u(x, t) \to 0$ as $x \to \infty$ and satisfies one of the boundary conditions

(a) $u = u_0$ at $x = 0$.

(b) $u_0 - u = h \dfrac{\partial u}{\partial x}$ at $x = 0$.

We want to determine either

(a) What is the temperature gradient at the origin, $u_x(0, t)$?

(b) What is the temperature at the origin, $u(0, t) = u_1(t)$?

Laplace Transforms: PDE to ODE

Defining the Laplace transform,

$$U(x,s)=\int_0^\infty u(x,t)e^{-st}\,dt,$$

and transforming the heat equation, we have

$$U_{xx}(x,s)-ksU(x,s)=0.$$

Bound solutions of this differential equation on $x\in[0,\infty)$ are given in the form

$$U(x,s)=A(s)e^{-\sqrt{ks}x}. \tag{27}$$

Laplace Transforms: Boundary Condition (a)

For boundary condition $u = u_0$ at $x = 0$ find $A(s)$.

We have $U(0, s) = \frac{u_0}{s} = A(s)$. Therefore,

$$U(x, s) = \frac{u_0}{s} e^{-\sqrt{ks}x}.$$

Since we want the gradient of the temperature at the origin,

$$U_x(0, s) = -\frac{u_0}{s}\sqrt{ks} = -\frac{u_0\sqrt{k}}{\sqrt{s}}.$$

The inverse Laplace transform of $s^{-1/2}$ from a table or inverse transform, we have

$$u_x(0, t) = -\frac{u_o\sqrt{k}}{\sqrt{\pi t}}.$$

Laplace Transforms: Boundary Condition (b)

For boundary condition $u_0 - u = h\frac{\partial u}{\partial x}$ at $x = 0$, its Laplace transform is

$$\frac{u_0}{s} - U(x,s) = h\frac{\partial U(x,s)}{\partial x}.$$

Inserting the solution $U(x,s) = A(s)e^{-\sqrt{ks}x}$ from Equation (27), we have

$$\frac{u_0}{s} - A(s)e^{-\sqrt{ks}x} = -h\sqrt{ks}A(s)e^{-\sqrt{ks}x}.$$

We now set $x = 0$ and solve for $A(s)$, to find

$$A(s) = \frac{u_0}{s(1 - h\sqrt{ks})}.$$

Laplace Transforms: Boundary Condition (b)

The Laplace transform of the solution to the boundary value problem is

$$U(x,s) = \frac{u_0}{s(1 - h\sqrt{ks})} e^{-\sqrt{ks}x}$$

and the Laplace transform of the solution at $x = 0$ is

$$U(0,s) = \frac{u_0}{s(1 - h\sqrt{ks})}. \tag{28}$$

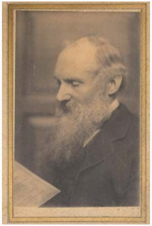
We can use a computer algebra system (CAS) like Maple or Mathematica. Doing so yields

$$u(0,t) = u_0 \left(1 - e^{\frac{t}{h^2 k}} \operatorname{erfc}\left(-\frac{\sqrt{t}}{h\sqrt{k}}\right)\right). \tag{29}$$

This solution agrees with Heaviside's solution (20) for $a = kh^2$.

Summary - Lessons from the Past

- Underwater Telegraphy
- Fourier's Heat Equation
- William Thomson/Lord Kelvin
- Oliver Heaviside
- Operational Calculus
- Laplace Transforms
- Part of upcoming book on *Applications of the Laplace Transform*



Thanks for Listening

Thank you for your attention.

References are provided on the remaining slides.

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