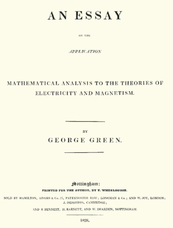
Why Are Green's Functions Used By Oil Companies and What is a Green's Function?

September 23, 2022 - Dr. R. L. Herman



AN ESSAY

ON THE

APPLICATION

MATHEMATICAL ANALYSIS TO THE THEORIES OF ELECTRICITY AND MAGNETISM.

BY

GEORGE GREEN.

Nottingham:

PRINTED FOR THE AUTHOR, BY T. WHEELHOUSE.

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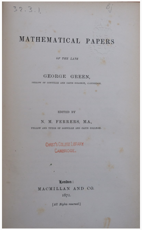
1828.



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MATHEMATICAL PAPERS

OF THE LATE

GEORGE GREEN,

FELLOW OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE.

EDITED BY

N. M. FERRERS, M.A.,

FELLOW AND TUTOR OF GONVILLE AND CAIUS COLLEGE.

London:

MACMILLAN AND CO.

1871.

Frame 1

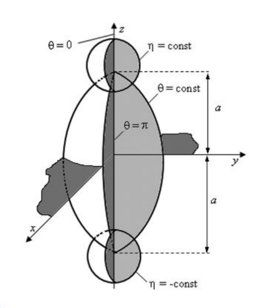
Introduction
George Green (1793-1841)
Green's Theorem
Green's Function
Cylindrical Oil Reservoir

Abstract

Green's functions, named after a relatively unknown grain miller, are used to solve boundary value problems. We will follow their path from an unknown 1928 Essay on the Application of Mathematical Analysis to the Theory of Electricity and Magnetism to a standard technique in mathematical physics in the mid-1900's. We sketch how Green's functions provide solutions to flow problems in reservoirs.

A Lifetime with Green's Functions

- Graduate School
  - Mathematical Physics
  - Electrodynamics
  - Integral Equations
  - Poisson-Boltzmann Equation between two spheres: $\nabla^2\psi = \sinh\psi$.
- Teaching
  - Differential Equations
  - Textbooks and Notes
- Recent - Petroleum Engineering



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The two-dimensional Laplace operator, ∇^2, in bispherical coordinates is given by:

$$\nabla^2\psi = \frac{(\cosh\eta - \cos\theta)^3}{\beta^2 \sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{\cosh\eta - \cos\theta}\frac{\partial\psi}{\partial\theta}\right) + \sin(\theta)\frac{\partial}{\partial\eta}\left(\frac{1}{\cosh\eta - \cos\theta}\frac{\partial\psi}{\partial\eta}\right)\right]$$

Who was Green?

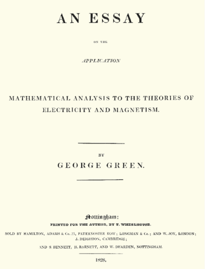
George Green's Life (1793-1841)

- Self-taught, about one year of formal schooling, between 8 and 9.
- Lived most of his life Sneinton, Nottinghamshire.
- His father, George, was a baker who built and owned a brick windmill to grind grain.
- In 1828 he published his famous essay.
  He published privately at his expense.
  Sold to 51 mostly friends.
- Wealthy landowner, mathematician, Edward Bromhead bought a copy and encouraged Green.
- Green did not contact Bromhead for two years.
- 1829, father died wealthy left to son and daughter.
- Younger George had time to pursue mathematics.
- In his final years at Cambridge, Green became rather ill, and in 1840 he returned to Sneinton, only to die a year later.

Discovery of 1828 Paper

- Green's work was not well known during his lifetime.
- In 1833 Robert Murphy (1806–1843) quoted the essay.
- In 1845, Green's essay was rediscovered & popularised by William Thomson (21 yr).
- In 1871 Ferrers assembled *The Mathem. Papers of the Late George Green*[a].
- Other contributions in these papers:

    On the motion of waves in a canal
    Green's pre-WKB approx.
    Green's Theorem and Identities.
    Green's Functions (named by Riemann), and potential functions.



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[a] His essay: arXiv:0807.0088.

Green's Theorem - Page 23, Green's *Essay*

Let U and V be two continuous functions of the rectangular co-ordinates x, y, z, whose differential co-efficients do not become infinite at any point within a solid body of any form whatever; then will

$$\int dxdydz\, U\delta V + \int d\sigma U \left(\frac{dV}{dw}\right) = \int dxdydz\, V\delta U + \int d\sigma V \left(\frac{dU}{dw}\right);$$

the triple integrals extending over the whole interior of the body, and those relative to $d\sigma$, over its surface, of which $d\sigma$ represents an element: dw being an infinitely small line perpendicular to the surface, and measured from this surface towards the interior of the body.

Modern Notation: Also *Green's Second Identity*

$$\int\int\int U\nabla^2 V\, dv + \int\int U\frac{\partial V}{\partial n}\, d\sigma = \int\int\int V\nabla^2 U\, dv + \int\int V\frac{\partial U}{\partial n}\, d\sigma.$$

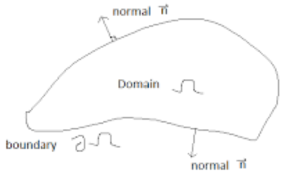
Equivalent to the Divergence Theorem. [Lagrange, 1754; Gauss, 1813 pub 1833,39; Ostrogradsky proved 1831.]

Green's Identities

In modern notation Green's first and second identities are given by the following with ψ and ϕ satisfying appropriate conditions of differentiability on a given domain..

$$\int_\Omega [\phi\nabla^2\psi + \nabla\psi\cdot\nabla\phi]\, dV = \int_{\partial\Omega} \phi\nabla\psi\cdot\mathbf{n}\, dS. \tag{1}$$

$$\int_\Omega [\phi\nabla^2\psi - \psi\nabla^2\phi]\, dV = \int_{\partial\Omega} [\phi\nabla\psi - \psi\nabla\phi]\cdot\mathbf{n}\, dS. \tag{2}$$



Poisson's Equation - Green Introduced in Essay

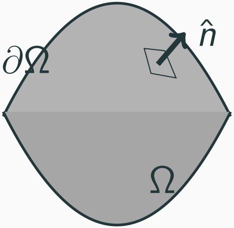
Let Poisson's equation,

$$\nabla^2 u(\mathbf{r}) = f(\mathbf{r}),$$

hold inside a region Ω bounded by the surface $\partial\Omega$.

This is the nonhomogeneous form of Laplace's equation.

$f(\mathbf{r})$, could represent a heat source in a steady-state problem, a charge distribution in an electrostatic problem, or an oil well.



Green's Function - Modern Approach

What is the response of the system to a point source?

The point source at $\mathbf{r}'$ is felt at $\mathbf{r}$. Call the response $G(\mathbf{r}, \mathbf{r}')$.

The response (Green's) function would satisfy

$$\nabla^2 G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'),$$

where $\delta(\mathbf{r} - \mathbf{r}')$ is the Dirac delta function satisfying

$$\delta(\mathbf{r}) = 0, \quad \mathbf{r} \neq \mathbf{0},$$

$$\int_\Omega \delta(\mathbf{r})\, dV = 1.$$

$$\int_\Omega \delta(\mathbf{r} - \mathbf{r}') f(\mathbf{r})\, dV = f(\mathbf{r}').$$

Green and others talked about *singularity.* It wasn't until mid-1900's that the Dirac delta function was understood in theory of distributions.

Green's Function and Poisson's Equation, $\nabla^2 u = 0$.

Green's second identity:

$$\int_{\partial\Omega} [\phi\nabla\psi - \psi\nabla\phi] \cdot \mathbf{n}\, dS = \int_{\Omega} [\phi\nabla^2\psi - \psi\nabla^2\phi]\, dV.$$

Letting $\phi = u(\mathbf{r})$ and $\psi = G(\mathbf{r}, \mathbf{r}')$, we have

$$\begin{aligned}
& \int_{\partial\Omega} [u(\mathbf{r})\nabla G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}')\nabla u(\mathbf{r})] \cdot \mathbf{n}\, dS \\
= & \int_{\Omega} \left[u(\mathbf{r})\nabla^2 G(\mathbf{r}, \mathbf{r}') - G(\mathbf{r}, \mathbf{r}')\nabla^2 u(\mathbf{r})\right]\, dV \\
= & \int_{\Omega} [u(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}') - G(\mathbf{r}, \mathbf{r}')f(\mathbf{r})]\, dV \\
= & \ u(\mathbf{r}') - \int_{\Omega} G(\mathbf{r}, \mathbf{r}')f(\mathbf{r})\, dV. \qquad (3)
\end{aligned}$$

Solve for $u(\mathbf{r}')$.

Solution

We have

$$\begin{aligned} u(\mathbf{r}') &= \int_\Omega G(\mathbf{r},\mathbf{r}') f(\mathbf{r})\, dV \\ &\quad + \int_{\partial\Omega} [u(\mathbf{r})\nabla G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\nabla u(\mathbf{r})] \cdot \mathbf{n}\, dS. \qquad (4) \end{aligned}$$

If both $u(\mathbf{r})$ and $G(\mathbf{r},\mathbf{r}')$ satisfied Dirichlet conditions, $u = 0$ on $\partial\Omega$, then the last integral vanishes and we are left with

$$u(\mathbf{r}') = \int_\Omega G(\mathbf{r},\mathbf{r}') f(\mathbf{r})\, dV.$$

In many applications, $G(\mathbf{r},\mathbf{r}') = G(\mathbf{r}',\mathbf{r})$. Then,

$$u(\mathbf{r}) = \int_\Omega G(\mathbf{r},\mathbf{r}') f(\mathbf{r}')\, dV'.$$

If we know the Green's function, we can solve nonhomogeneous differential equations and nonhomogenous boundary value and initial value problems.

Green's Functions for PDE's

The wave, heat, and Laplace's equation are typical homogeneous PDEs.

$$\begin{aligned}\left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right) u &= 0,\\ \left(\frac{\partial}{\partial t} - k\nabla^2\right) u &= 0,\\ \nabla^2 u &= 0. \end{aligned} \tag{5}$$

They can be written in the form

$$\mathcal{L}u(x) = 0,$$

where $\mathcal{L}$ is a differential operator, $x = \{\mathbf{r}, t\}$.

We solve the nonhomogeneous equations, $\mathcal{L}u(x) = f(x)$, by seeking out the Green's function, $\mathcal{L}G(x, x') = \delta(x - x')$.

Diffusion Equation

$$\frac{\partial u(\mathbf{r},t)}{\partial t} - k\nabla^2 u(\mathbf{r},t) = S(\mathbf{r},t).$$

$$\frac{\partial G(\mathbf{r},\mathbf{r}',t,t')}{\partial t} - k\nabla^2 G(\mathbf{r},\mathbf{r}',t,t') = \delta(\mathbf{r}-\mathbf{r}')\delta(t-t').$$

How to find $G(\mathbf{r},\mathbf{r}',t,t')$:

Integral transforms: Laplace to remove t-dependence.

Symmetry to reduce to ODEs.

Often, one obtains Green's functions analytically as infinite series.
Problems with convergence.

There are numerical techniques needed for inverse Laplace transforms.

Cylindrical Oil Reservoirs - Diffusion Equation

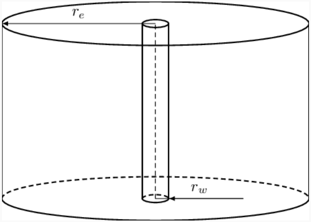
Let $p = p(r,t)$ be the pressure in the reservior. A general form of the line source problem would be

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) - \frac{\partial p}{\partial t} = S(r,t), \quad (6)$$

$$p(r,0) = p_0(r),$$

$$\lim_{r\to 0} r\frac{\partial p}{\partial r} = b(t), \quad \lim_{r\to\infty} p(r,t) = 0. \quad (7)$$

Here $S(r,t)$ would be a source term and $b(t)$ a lower boundary value.



Cylindrical Oil Reservoirs - Boundary Conditions

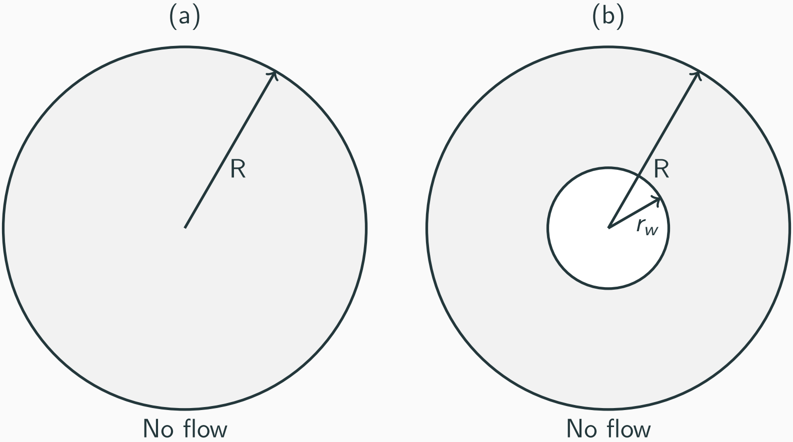


Figure 2: Examples with no flow boundaries. (a) Line source in a finite region of radius R. (b) Finite well source of radius r_w inside a finite region of radius R.

Laplace Transform

For the diffusion equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) - \frac{\partial p}{\partial t} = S(r,t), \tag{8}$$
$$p(r,0) = p_0(r),$$

one can apply the Laplace transform,

$$\hat{p}(r,s) = \int_0^\infty p(r,t)e^{-st}\,dt,$$

to the diffusion equation to obtain

$$r^2\frac{\partial^2 \hat{p}}{\partial r^2} + r\frac{\partial \hat{p}}{\partial r} - r^2 s\hat{p} = -r^2 p_0(r) + r^2\hat{S}(r,s) \equiv F(r,s).$$

The general solution takes the form

$$\hat{p}(r,s) = c_1 I_0(r\sqrt{s}) + c_2 K_0(r\sqrt{s}) + P(r,s). \tag{9}$$

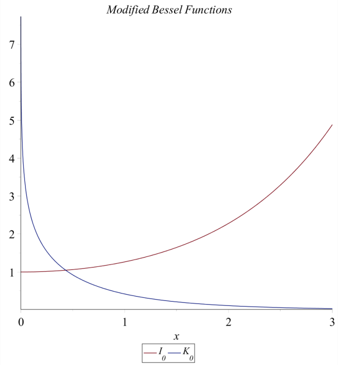
Modified Bessel Functions

$I_n(x)$: Modified Bessel function of the first kind.

$K_n(x)$: Modified Bessel function of the second kind.

These are related to the Bessel functions of the first and second kind,

$$I_n(x) = i^{-n} J_n(ix), \qquad K_n(x) = \frac{\pi}{2} \frac{I_{-n}(x) - I_n(x)}{\sin n\pi}.$$



Examples

At this point one considers different boundary value problems and if one can find exact solutions or numerical solutions.

To make sure numerical techniques work, one needs analytical solutions to toy problems.

- Solve for $\hat{p}(r,s)$.
- Find $p(r,t)$ using table look-up or the Bromwich integral.
  The Laplace transform of a function $f(t)$ is defined as

$$F(s) = \mathcal{L}[f](s) = \int_0^\infty f(t)e^{-st}\,dt, \quad s > 0. \tag{10}$$

  The inverse Laplace transform is obtained using the Bromwich integral, or Fourier-Mellin integral,

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)e^{st}\,ds. \tag{11}$$

Next - Some rough computations!

Next we go through a few notes indicating the type of functions and mathematics needed.

- Modified Bessel functions.
- Line source infinite reservoir.
- Constant rate at finite radius.
- No-flow boundaries.
- Finite wells.
- Application of Green's functions.
- Inversion of Laplace transform.

During the presentation some research notes were presented. Here we add the important computation of the inverse Laplace transform.

Inverse Laplace Transform

Starting with the Bromwich integral,

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)e^{st}\, ds, \tag{12}$$

we choose c so that all poles are to the left of the contour such as seen in Figure 3. The contour is a closed semicircle enclosing all the poles. One then relies on a generalization of Jordan's Lemma.[1]

We follow up with a seemingly simple problem.

---

[1]One has a choice to close the contour to the left or right of the contour. Writing the exponential as $e^{st} = e^{(s_R + is_I)t} = e^{s_R t} e^{is_I t}$, we see that the second factor is an oscillating factor. The growth in the exponential can only come from the first factor. In order for the exponential to decay as the radius of the semicircle grows, we need $s_R t < 0$. Since $t > 0$, then $s < 0$ and we close the contour to the left. If $t < 0$, then the enclosed contour to the right would enclose no singularities and preserve the causality of $f(t)$.

Inverse Laplace Transform - Contour

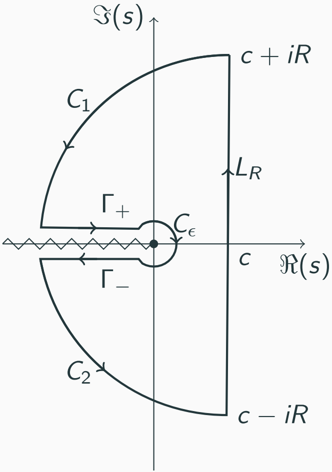


Figure 3: A contour used for applying the Bromwich integral to the Laplace transform $F(s) = K_0(r\sqrt{s})$ with a branch point at $s = 0$.

Inverse Laplace Transform - Example

Example

Let $F(s) = K_0(r\sqrt{s})$. Then, the Bromwich integral is given by

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} K_0(r\sqrt{s}) e^{st}\, ds. \tag{13}$$

Since there is a branch point at $s = 0$, we choose $c > 0$.

We consider the complex integral around the contour

$$C = \lim_{R\to\infty} \lim_{\epsilon\to 0} (L_R + C_1 + \Gamma_+ + C_\epsilon + \Gamma_- + C_2)$$

in Figure 3,

$$I = \frac{1}{2\pi i} \oint_C K_0(r\sqrt{z}) e^{tz}\, dz = 0.$$

Inverse Laplace Transform - Example (cont'd)

The integrals over C_1 and C_2 vanish by Jordan's Lemma. So, we need to compute $f(t)$ as

$$\lim_{R\to\infty}\lim_{\epsilon\to 0}\left(-\frac{1}{2\pi i}\int_{\Gamma_+} K_0(r\sqrt{z})e^{tz}\,dz - \frac{1}{2\pi i}\int_{C_\epsilon} K_0(r\sqrt{z})e^{tz}\,dz - \frac{1}{2\pi i}\int_{\Gamma_-} K_0(r\sqrt{z})e^{tz}\,dz\right).$$

First we note that

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{C_\epsilon} K_0(r\sqrt{z})e^{tz}\,dz &= -\lim_{\epsilon\to 0}\int_{-\pi}^{\pi} K_0(r\sqrt{\epsilon e^{i\theta}})e^{t\epsilon e^{i\theta}}\,i\epsilon e^{i\theta}\,d\theta \\ &= \lim_{\epsilon\to 0}\frac{i}{2}\epsilon\ln\epsilon\int_{-\pi}^{\pi} d\theta = 0. \qquad (14)\end{aligned}$$

For the integrals over $\Gamma_\pm$ we let $z = xe^{\pm i\pi}$, giving

$$\begin{aligned}\int_{\Gamma_\pm} K_0(r\sqrt{z})e^{tz}\,dz &= \mp\int_0^\infty K_0(r\sqrt{xe^{\pm i\pi}})e^{txe^{\pm i\pi}}e^{\pm i\pi}\,dx \\ &= \pm\int_0^\infty K_0(\pm ir\sqrt{x})e^{-tx}\,dx.\end{aligned}$$

Inverse Laplace Transform - Example (cont'd)

Letting $y = r\sqrt{x}$, or $x = y^2/r^2$ and $dx = 2ydy/r^2$, we have

$$\pm \int_0^\infty K_0(\pm ir\sqrt{x}) e^{-tx}\, dx = \pm \frac{2}{r^2} \int_0^\infty K_0(\pm iy) e^{-ty^2/r} y\, dy$$

Combining the integrals, we have

$$\begin{aligned} & \int_{\Gamma_+} K_0(r\sqrt{z}) e^{tz}\, dz + \int_{\Gamma_-} K_0(r\sqrt{z}) e^{tz}\, dz && (15) \\ = & \int_0^\infty [K_0(ir\sqrt{x}) - K_0(-ir\sqrt{x})] e^{-tx}\, dx \\ = & -i\pi \int_0^\infty J_0(r\sqrt{x}) e^{-tx}\, dx, && (16) \end{aligned}$$

where we employed the identity

$$K_0(ir\sqrt{x}) - K_0(-ir\sqrt{x}) = -i\pi I_0(ir\sqrt{x}) = -i\pi J_0(r\sqrt{x}), \quad r > 0.$$

Inverse Laplace Transform - Example (cont'd)

We need the Laplace transform of $J_0(r\sqrt{x})$. Writing $w(x) = J_0(r\sqrt{x})$, $w(x)$ satisfies the initial value problem

$$4xw'' + 4w' + r^2 w = 0, \quad w(0) = 1, w'(0) = 0.$$

Taking the Laplace transform, $W(t) = \int_0^\infty J_0(r\sqrt{x})e^{-tx}\,dx$,

$$\begin{aligned} 0 &= -4\frac{d}{dt}(t^2W(t) - t) + 4(tW(t) - 1) + r^2W(t) \\ &= -4t^2W'(t) - 4tW(t) + r^2W(t) \end{aligned} \tag{17}$$

The solution of this differential equation is $W(t) = \frac{1}{t}e^{-r^2/4t}$. Therefore, we have that

$$f(t) = \frac{1}{2t}e^{-r^2/4t}.$$

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