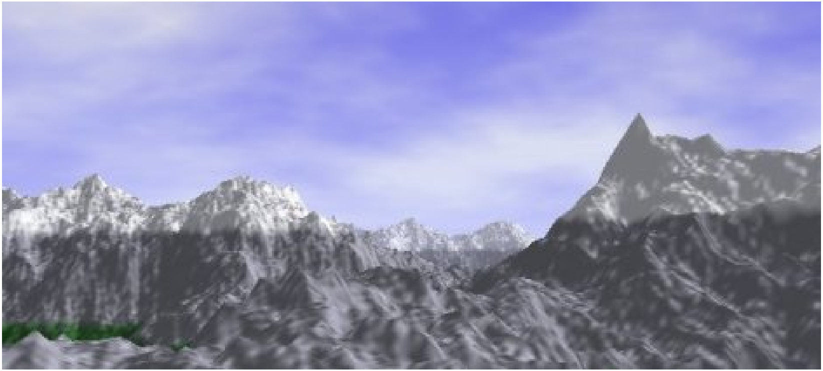
Butterflies, Ferns, and Fractal Landscapes: The Emergence of Complexity from Simple Systems

Dr. R. L. Herman, UNCW Mathematics/Physics



Outline

1. Butterflies
2. Fractals
3. Ferns
4. Fractal Landscapes

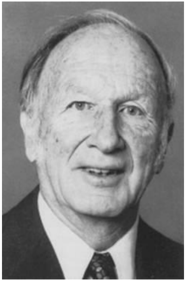
Ian Malcolm (Jeff Goldblum), *Jurassic Park* (1993)

"tiny variations, the orientation of hairs on your hand, the amount of blood distending your vessels ... vastly affect the outcome"
`https://www.youtube.com/watch?v=n-mpifTiPV4`

Edward Lorenz (1917-2008) - Weather Modeling

Fluid circulation in a shallow fluid layer,

- heated uniformly from below
- cooled uniformly from above



$$
\begin{aligned}
\frac{dx}{dt} &= \sigma(y-x)\\
\frac{dy}{dt} &= x(\rho - z) - y\\
\frac{dz}{dt} &= xy - \beta z
\end{aligned}
$$

"Two states differing by imperceptible amounts may eventually evolve into two considerably different states ....In view of the inevitable inaccuracy and incompleteness of weather observations, precise very-long-range forecasting would seem to be nonexistent." - 1963

Lorenz Model Solution

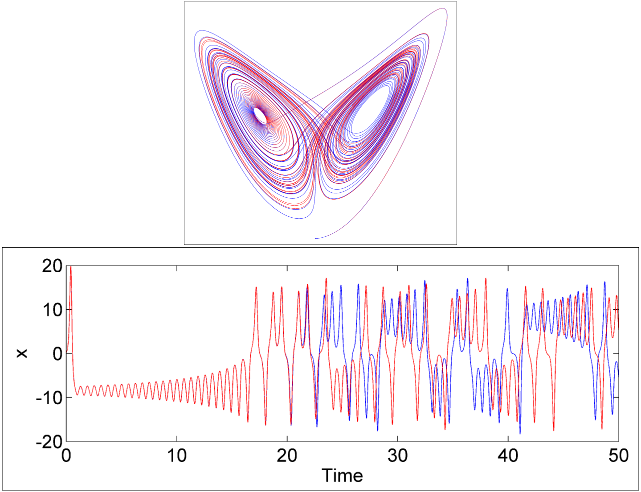


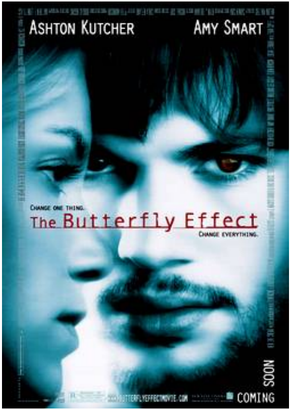
Figure : Solutions for $x(t)$ with initial conditions differing by 1%

http://logicaltightrope.com/2013/08/29/edward-lorenzs-strange-attraction/

Sea Gulls, Butterflies, and Grasshoppers

Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?, Lorenz, AAAS 1972

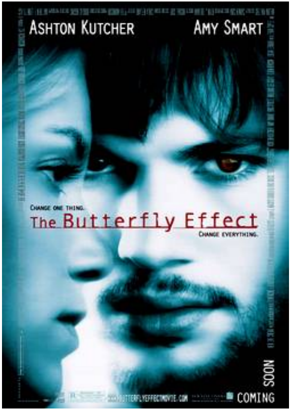
- "... one flap of a sea gull's wings would be enough to alter the course of the weather forever." - Lorenz, 1963



Sea Gulls, Butterflies, and Grasshoppers

Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?, Lorenz, AAAS 1972

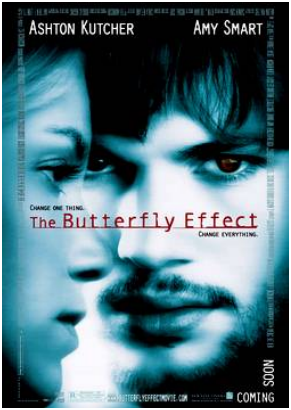
- "... one flap of a sea gull's wings would be enough to alter the course of the weather forever." - Lorenz, 1963
- "A dead mouse here makes an insect imbalance there, ... and, finally a change in social temperment in far-flung countries. ' 'Ray Bradbury, *A Sound of Thunder*, 1952



Sea Gulls, Butterflies, and Grasshoppers

Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?, Lorenz, AAAS 1972

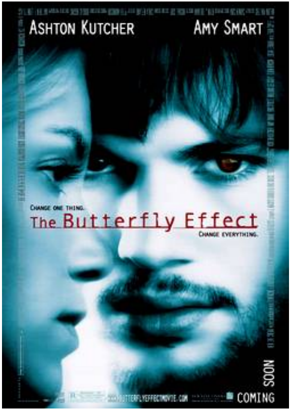
- "... one flap of a sea gull's wings would be enough to alter the course of the weather forever." - Lorenz, 1963
- "A dead mouse here makes an insect imbalance there, ... and, finally a change in social temperment in far-flung countries. ' 'Ray Bradbury, *A Sound of Thunder*, 1952
- "A Chinaman sneezing in Shen-si may set men shoveling snow in New York City ' 'George R. Stewart, *Storm*, 1941



Sea Gulls, Butterflies, and Grasshoppers

Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?, Lorenz, AAAS 1972

- "... one flap of a sea gull's wings would be enough to alter the course of the weather forever." - Lorenz, 1963
- "A dead mouse here makes an insect imbalance there, ... and, finally a change in social temperment in far-flung countries. ' 'Ray Bradbury, *A Sound of Thunder*, 1952
- "A Chinaman sneezing in Shen-si may set men shoveling snow in New York City ' 'George R. Stewart, *Storm*, 1941
- "... the flight of a grasshopper in Montana may turn a storm aside from Philadelphia to New York." W. S. Franklin, 1898



Henri Poincaré (1854-1912)

- Oscar II, King of Sweden offered birthday prize 1887
- Poincaré awarded for 3-body problem
- First description of chaos!

Is the Solar System Stable?

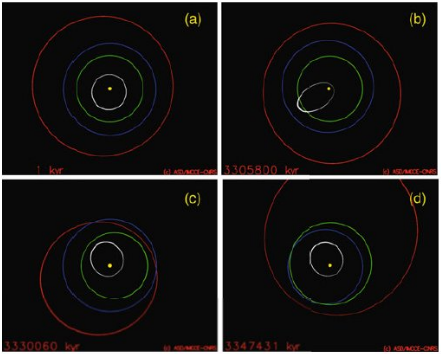


FIGURE 6. Example of long-term evolution of the orbits of the terrestrial planets: Mercury (white), Venus (green), Earth (blue), Mars (red). The time is indicated in thousands of years (kyr). (a) In the vicinity of the current state, the orbits are deformed under the influence of planetary perturbations, but without allowing close encounters or collisions.

Sensitivity to Initial Conditions

"If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. but even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation approximately. If that enabled us to predict the succeeding situation with the same approximation, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by laws. But it is not always so; **it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter.** Prediction becomes impossible, and we have the fortuitous phenomenon." - Henri Poincaré (1903), *Science and Method*

Logistic Map - $x_{n+1} = rx_n(1 - x_n)$, given x_0

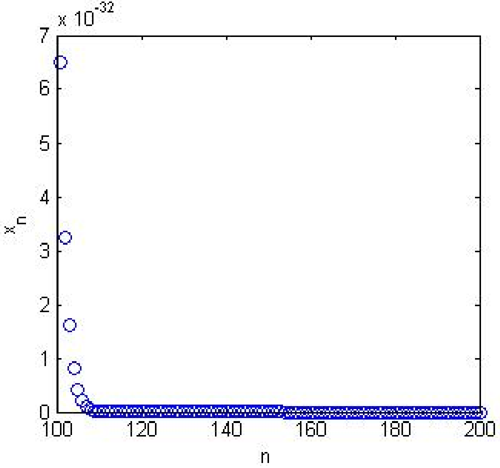


Figure : $r = 0.05$

Logistic Map - $x_{n+1} = rx_n(1 - x_n)$, given x_0

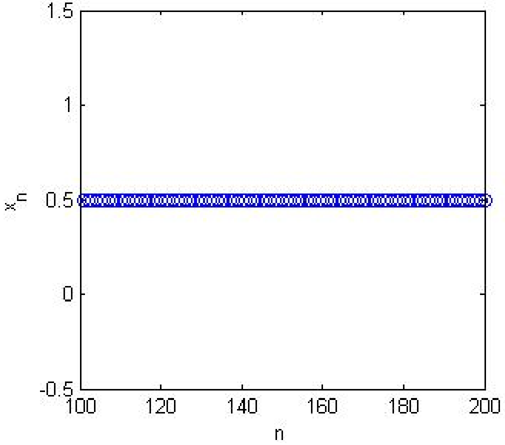


Figure : $r = 2.0$

Logistic Map - $x_{n+1} = rx_n(1 - x_n)$, given x_0

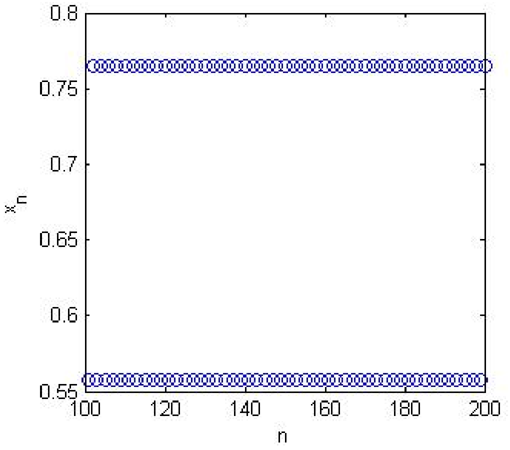


Figure : $r = 3.1$

Logistic Map - $x_{n+1} = rx_n(1 - x_n)$, given x_0

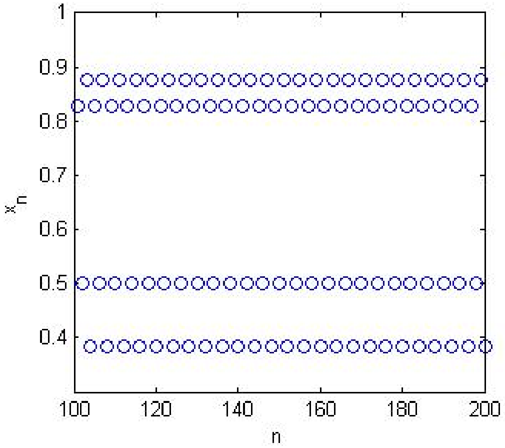


Figure : $r = 3.5$

Logistic Map - $x_{n+1} = rx_n(1 - x_n)$, given x_0

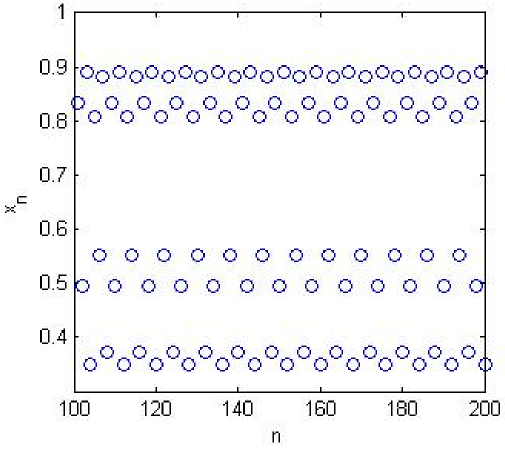


Figure : $r = 3.56$

Logistic Map - $x_{n+1} = rx_n(1 - x_n)$, given x_0

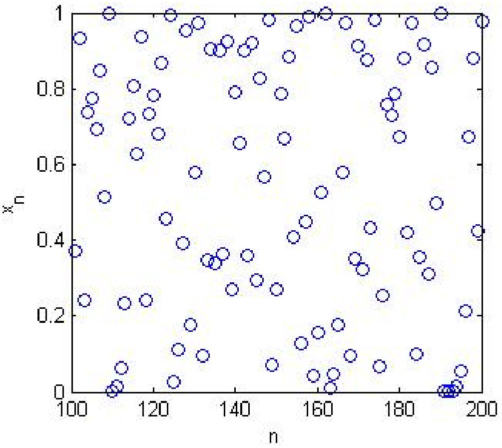
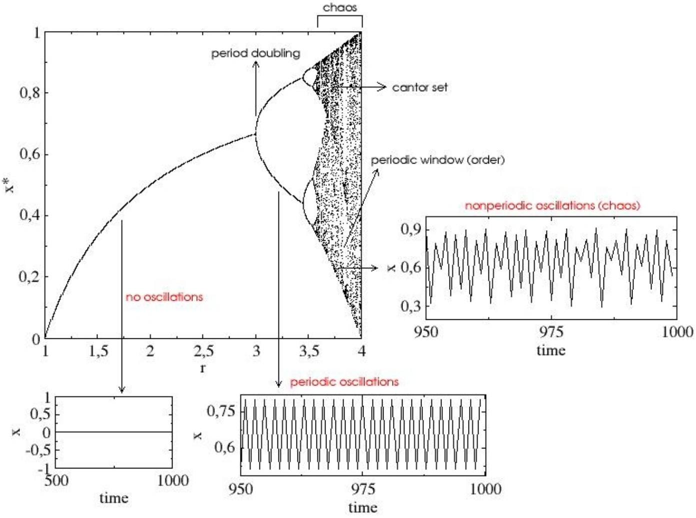
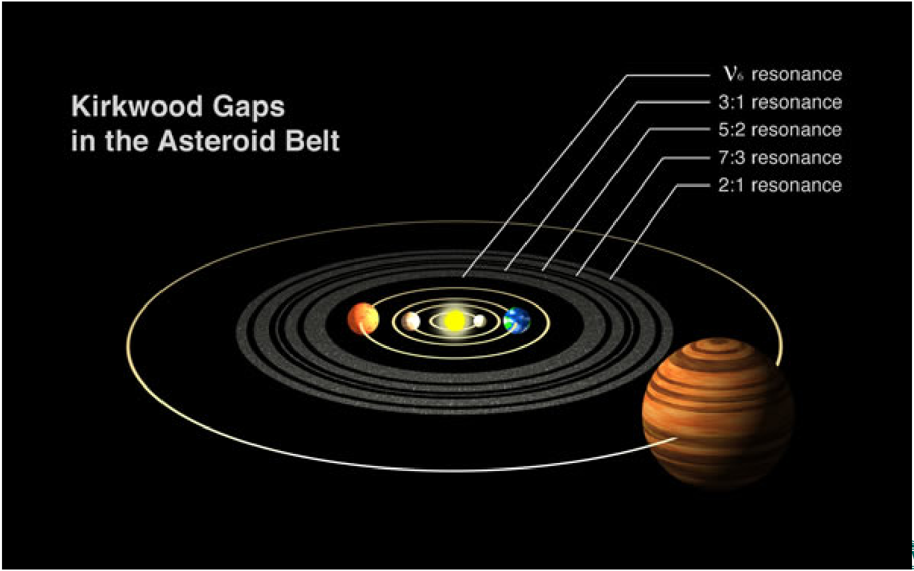


Figure : $r = 4.0$

Bifurcations of Logistic Map - $x_{n+1} = rx_n(1 - x_n)$, given x_0

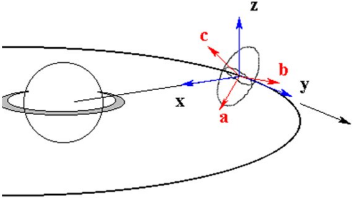


Asteroids



Kirkwood Gaps
in the Asteroid Belt
ν_6 resonance
3:1 resonance
5:2 resonance
7:3 resonance
2:1 resonance

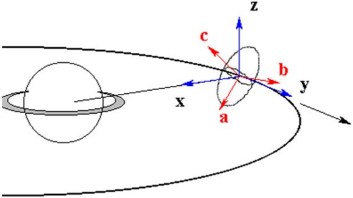
Hyperion - Moon of Saturn



- Only moon rotating chaotically

$$
\begin{aligned}
\frac{d^2\theta}{dt^2} &= -\frac{GM_S}{2r^3}\omega_0^2 \sin 2(\theta - \phi), \\
r &= \frac{a(1-e)}{1+e\cos\phi} \\
\omega_0^2 &= \frac{3(B-A)}{C}.
\end{aligned}
$$

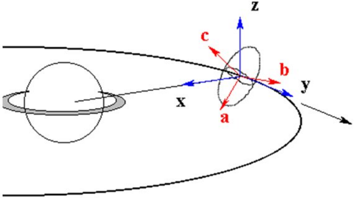
Hyperion - Moon of Saturn



- Only moon rotating chaotically
- Axis of rotation wobbles

$$\frac{d^2\theta}{dt^2} = -\frac{GM_S}{2r^3}\omega_0^2 \sin 2(\theta - \phi),$$
$$r = \frac{a(1-e)}{1+e\cos\phi}$$
$$\omega_0^2 = \frac{3(B-A)}{C}.$$

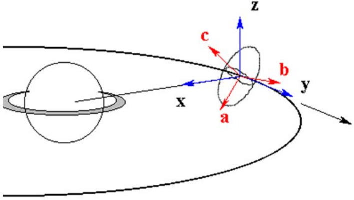
Hyperion - Moon of Saturn



- Only moon rotating chaotically
- Axis of rotation wobbles
- Irregular shape (tidal torque)

$$\begin{aligned}\frac{d^2\theta}{dt^2} &= -\frac{GM_S}{2r^3}\omega_0^2 \sin 2(\theta - \phi),\\ r &= \frac{a(1-e)}{1+e\cos\phi}\\ \omega_0^2 &= \frac{3(B-A)}{C}.\end{aligned}$$

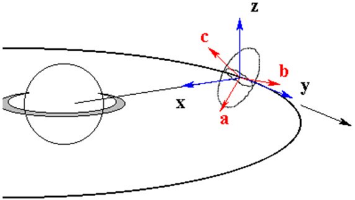
Hyperion - Moon of Saturn



- Only moon rotating chaotically
- Axis of rotation wobbles
- Irregular shape (tidal torque)
- Eccentric orbit ($e = 0.123$)

$$\begin{aligned}
\frac{d^2\theta}{dt^2} &= -\frac{GM_S}{2r^3}\omega_0^2 \sin 2(\theta - \phi), \\
r &= \frac{a(1-e)}{1+e\cos\phi} \\
\omega_0^2 &= \frac{3(B-A)}{C}.
\end{aligned}$$

Hyperion - Moon of Saturn



- Only moon rotating chaotically
- Axis of rotation wobbles
- Irregular shape (tidal torque)
- Eccentric orbit ($e = 0.123$)
- Resonance with Titan

$$\begin{aligned} \frac{d^2\theta}{dt^2} &= -\frac{GM_S}{2r^3}\omega_0^2 \sin 2(\theta - \phi), \\ r &= \frac{a(1-e)}{1 + e\cos\phi} \\ \omega_0^2 &= \frac{3(B-A)}{C}. \end{aligned}$$

Hyperion Dynamics

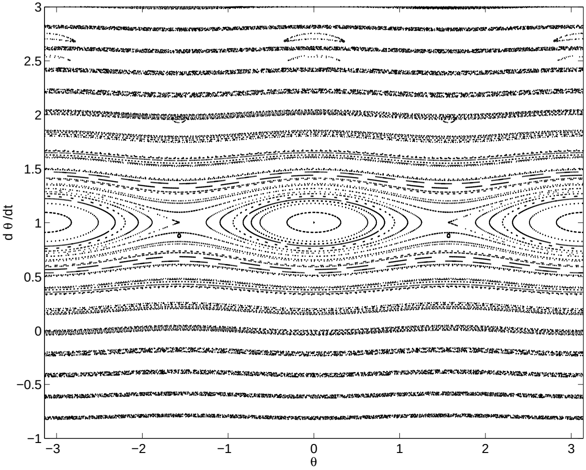


Figure : $\epsilon = 0, \omega_0 = 0.3$

Hyperion Dynamics

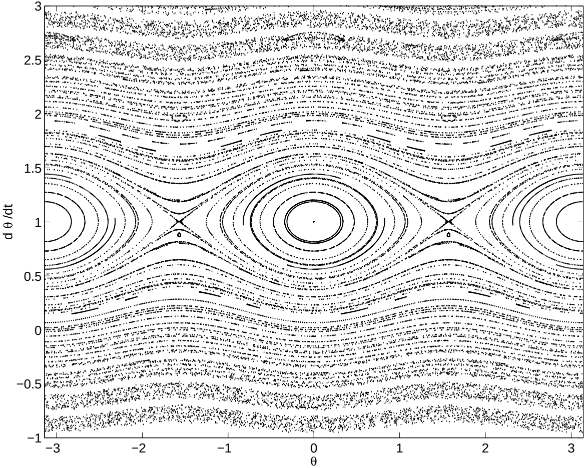


Figure : $\epsilon = 0, \omega_0 = 0.6$

Hyperion Dynamics

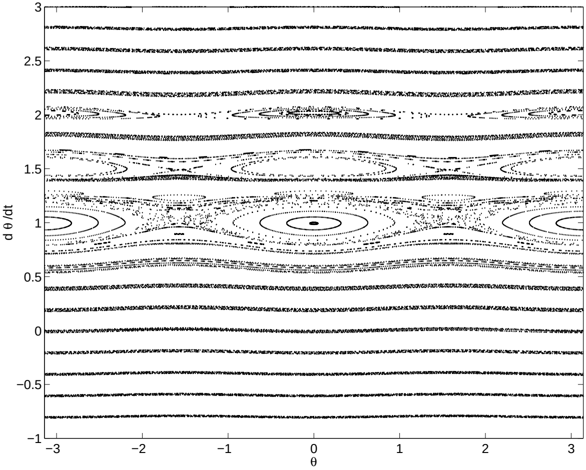


Figure : $\epsilon = 0.1, \omega_0 = 0.2$

Hyperion Dynamics

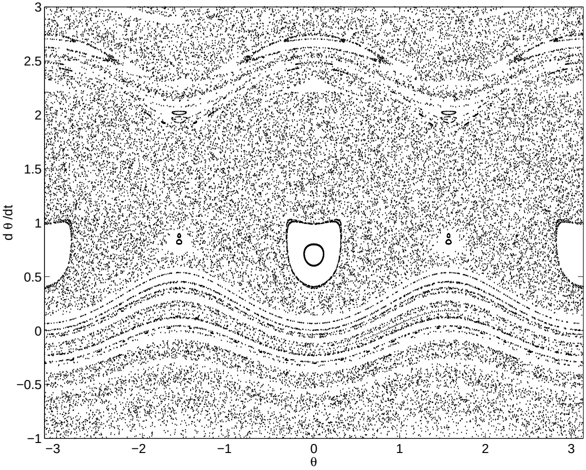


Figure : $\epsilon = 0.1, \omega_0 = 0.8$

Fractals

- *fractus*, - “broken”
- Self-similarity
- Dimension - Not an integer

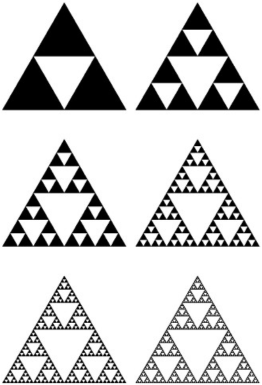


Figure : Sierpinski Triangle - 1915

Fractal Trees

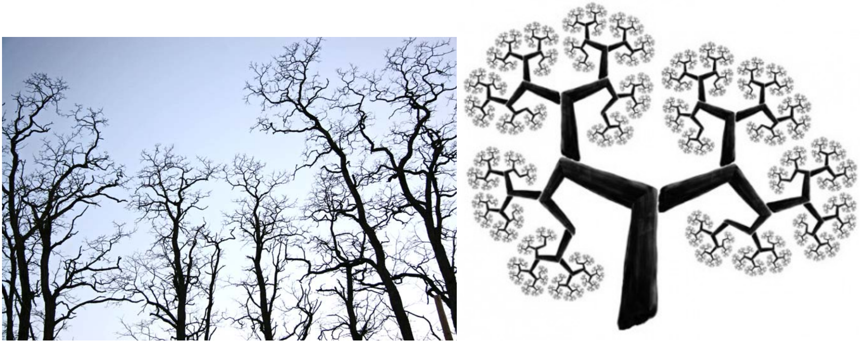


Figure : Fractals - self-similarity, roughness

Fractals in Nature

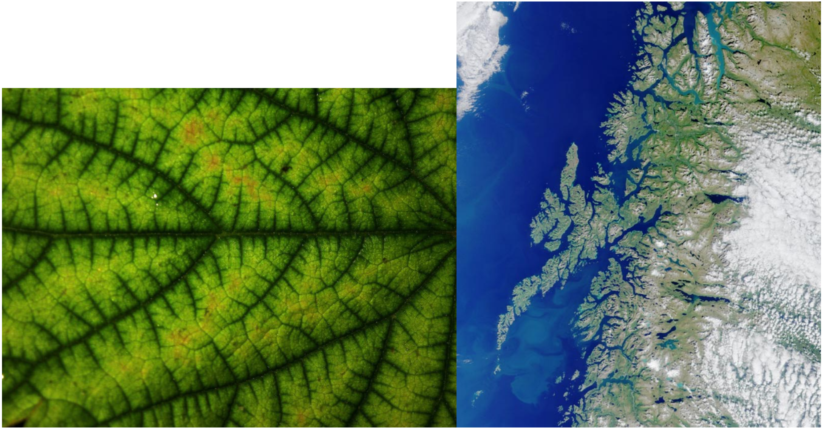
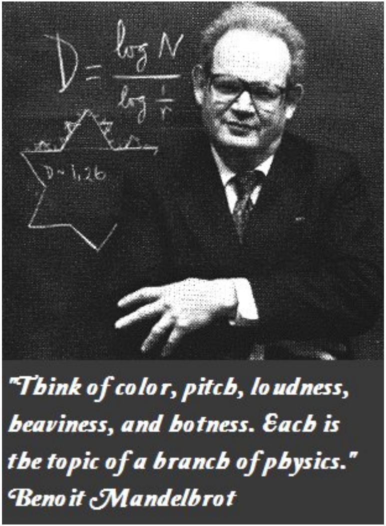


Figure : Fractals -what do you see?

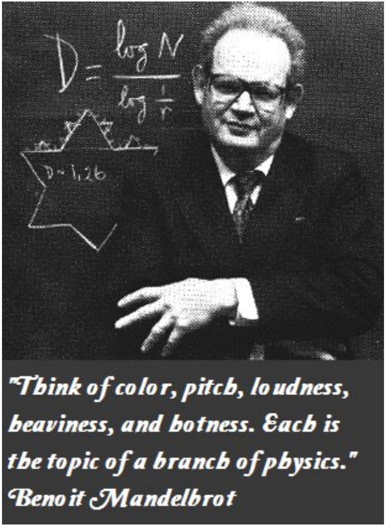
Benoît Mandelbrot (1924-2010)

- Grew up in France



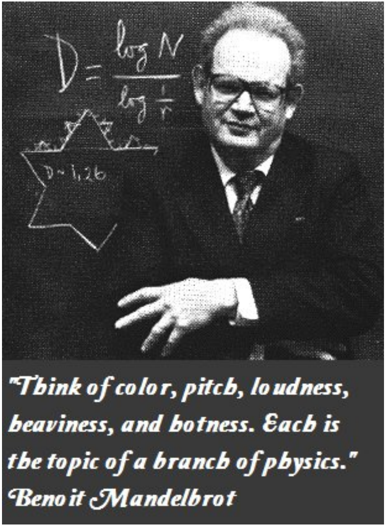
Benoît Mandelbrot (1924-2010)

- Grew up in France
- Paris and Caltech Education



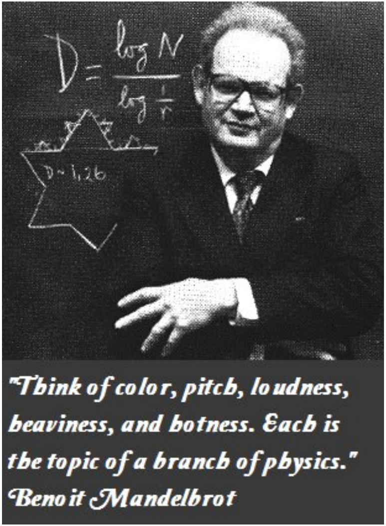
Benoît Mandelbrot (1924-2010)

- Grew up in France
- Paris and Caltech Education
- IBM Fellow



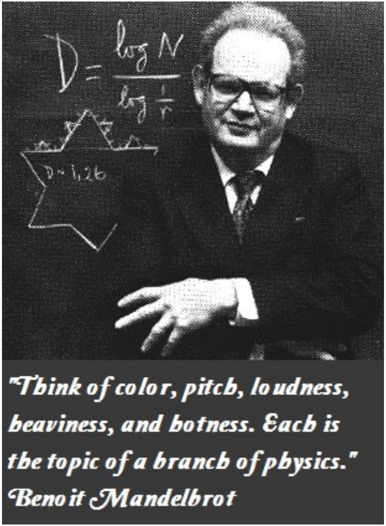
Benoît Mandelbrot (1924-2010)

- Grew up in France
- Paris and Caltech Education
- IBM Fellow
- Fractals



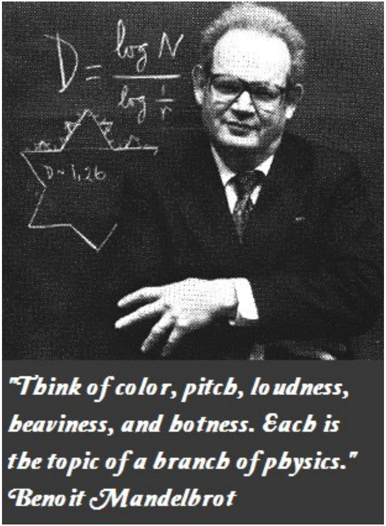
Benoît Mandelbrot (1924-2010)

- Grew up in France
- Paris and Caltech Education
- IBM Fellow
- Fractals
- Studied "roughness" in nature



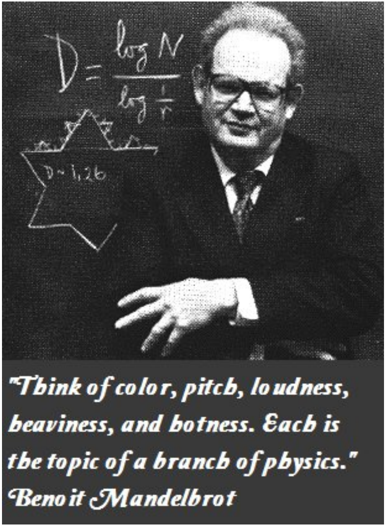
Benoît Mandelbrot (1924-2010)

- Grew up in France
- Paris and Caltech Education
- IBM Fellow
- Fractals
- Studied "roughness" in nature
- *Fractal Geometry*



Benoît Mandelbrot (1924-2010)

- Grew up in France
- Paris and Caltech Education
- IBM Fellow
- Fractals
- Studied “roughness” in nature
- *Fractal Geometry*
- Mandelbrot Set

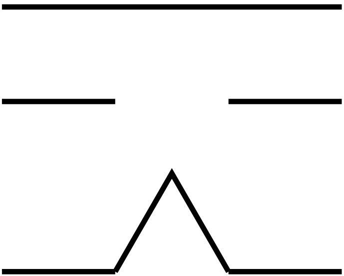


Simple Fractals - Koch Curve (1904)

Step 1 ($L = 1$)

Simple Fractals - Koch Curve (1904)

Simple Fractals - Koch Curve (1904)

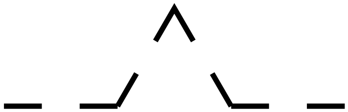


Step 1 ($L = 1$)

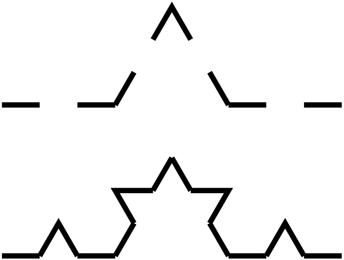
Step 2

Step 3 ($L = 4(\frac{1}{3}) = \frac{4}{3}$)

Simple Fractals - Koch Curve (1904)



Simple Fractals - Koch Curve (1904)



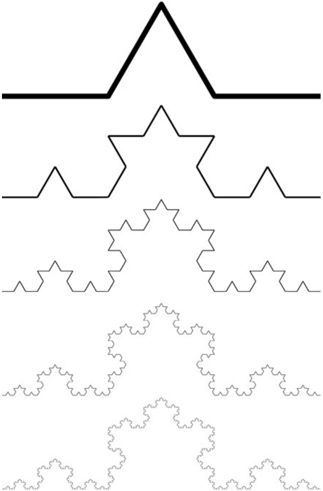
Step 1 ($L = 1$)

Step 2

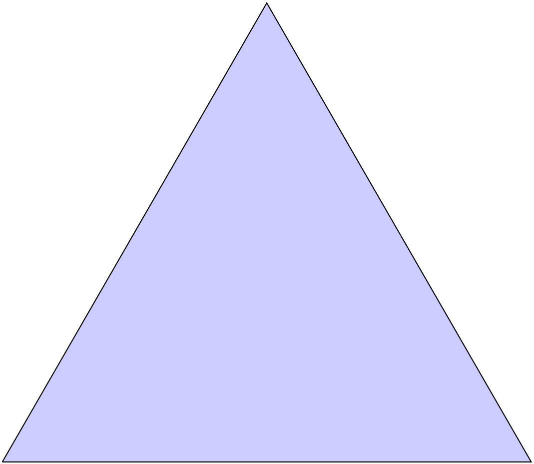
Step 4

Step 5 ($L = \frac{16}{9}$)

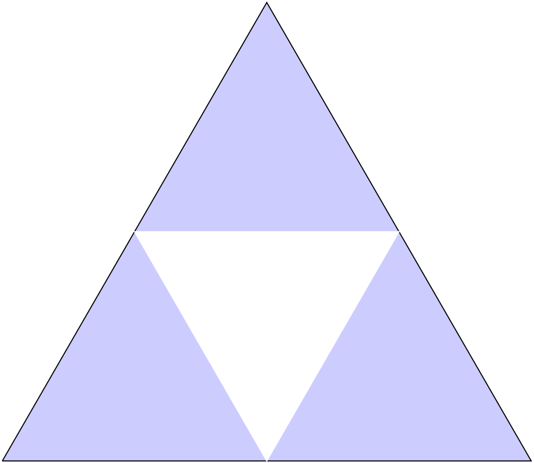
Koch Curve - Self Similarity ($L = \frac{4^n}{3^n} \to \infty$)



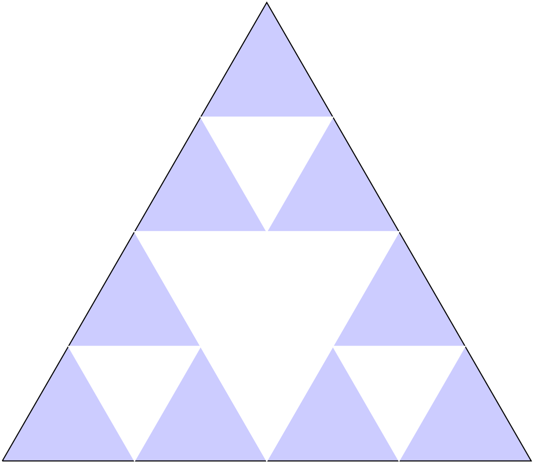
Simple Fractals - Sierpinski Triangle



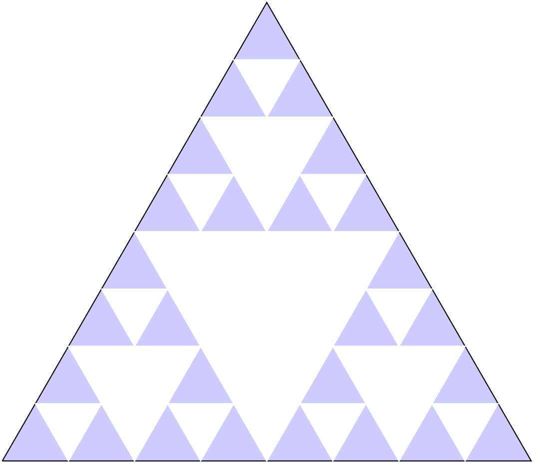
Simple Fractals - Sierpinski Triangle



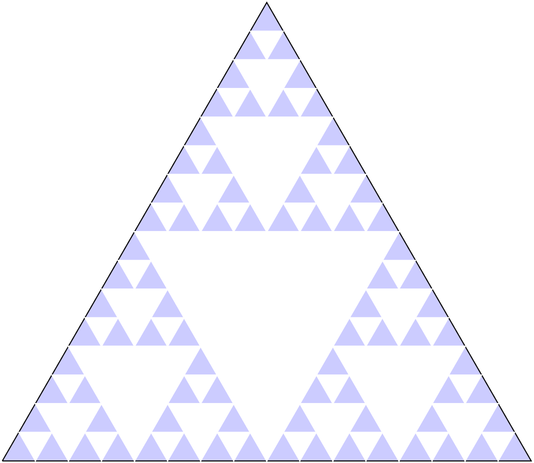
Simple Fractals - Sierpinski Triangle



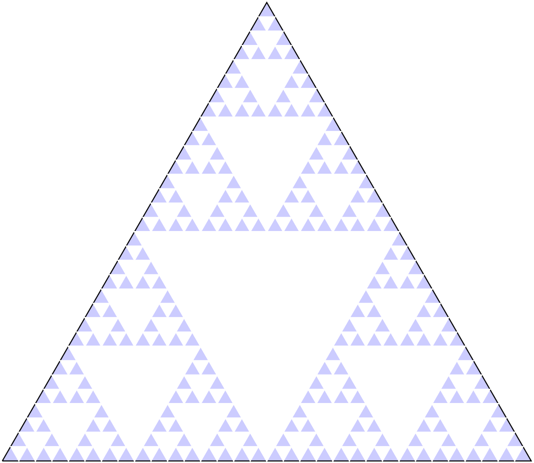
Simple Fractals - Sierpinski Triangle



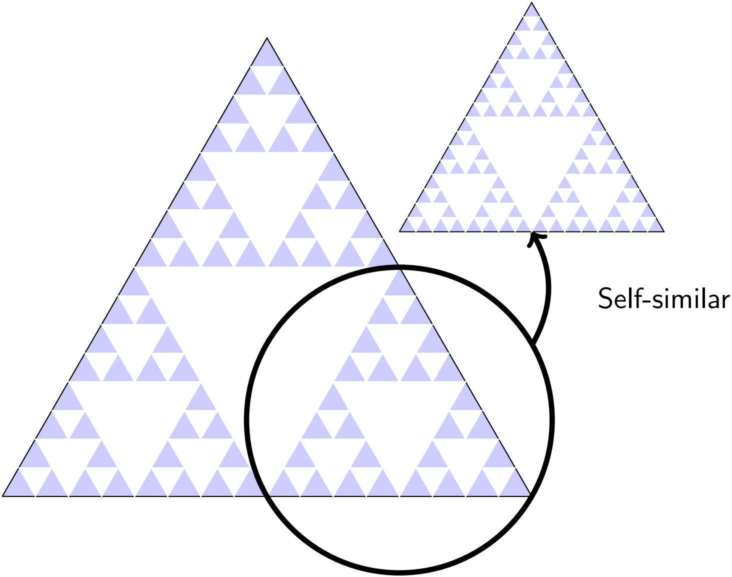
Simple Fractals - Sierpinski Triangle



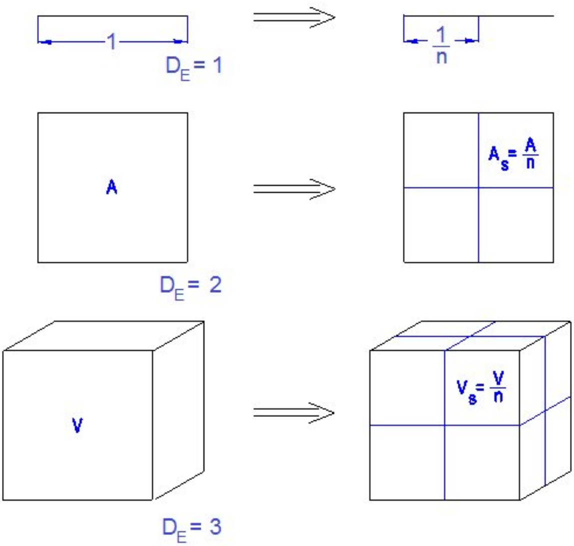
Simple Fractals - Sierpinski Triangle



Simple Fractals - Sierpinski Triangle



Dimensions - $r =$ magnification, $n =$ Number of shapes



$n = 2, \quad r = 2$

$\Rightarrow \ln 2 = \ln 2.$

$n = 4, \quad r = 2$

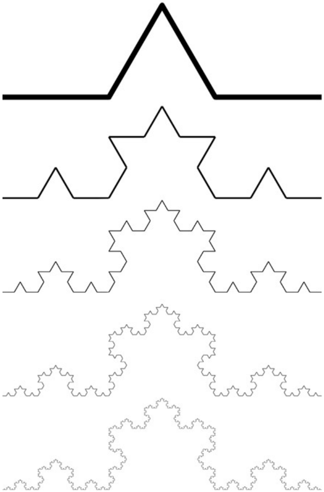
$\Rightarrow \ln 4 = 2 \ln 2.$

$n = 8, \quad r = 2$

$\Rightarrow \ln 8 = 3 \ln 2.$

$$\boxed{D = \frac{\ln n}{\ln r}.}$$

Fractal Dimensions - Koch Curve



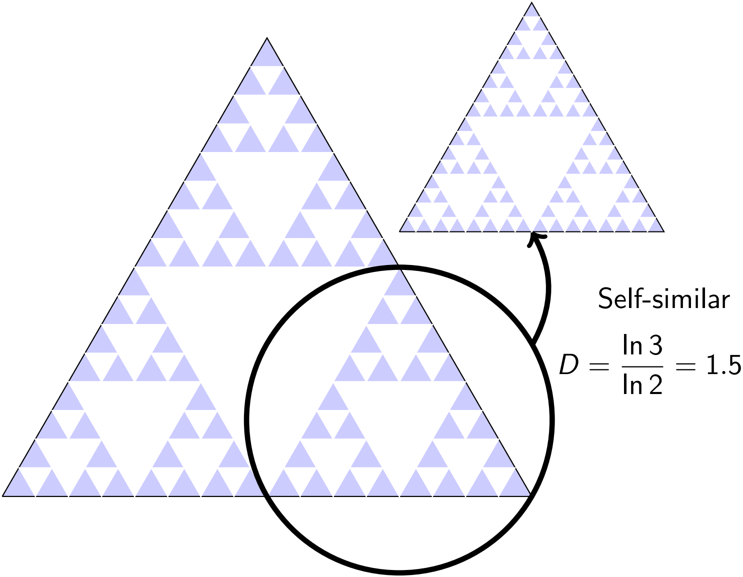
For each step length of line segment is reduced by $r = 3$.

The number of lines increases by factor $n = 4$.

Therefore

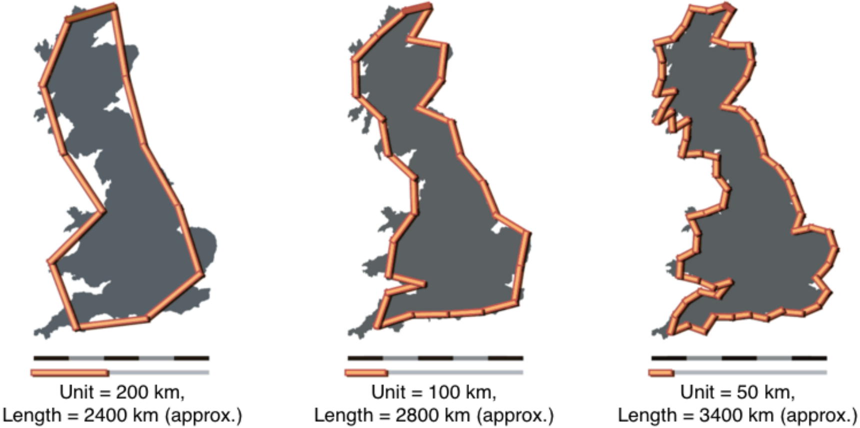
$$\begin{aligned} D &= \frac{\ln n}{\ln r} \\ &= \frac{\ln 4}{\ln 3} \\ &= 1.26. \end{aligned}$$

Sierpinski Triangle - Dimension



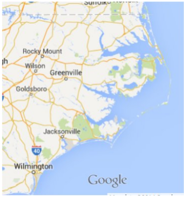
$$D = \frac{\ln 3}{\ln 2} = 1.585$$

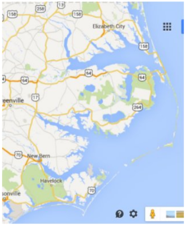
Coastlines - Great Britain, $D = 1.25$

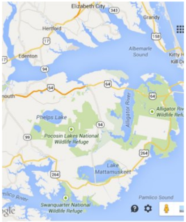


Coastlines -North Carolina

What is the fractal dimension of the NC coast?







Mandelbrot Set

Iterate complex numbers
$z = a + bi$, $i = \sqrt{-1}$.

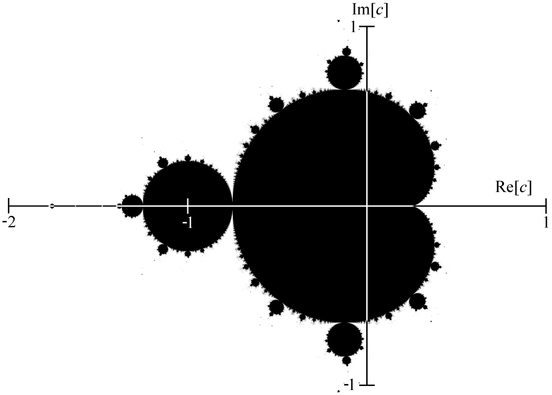
$$z_{n+1} = z_n^2 + c, \quad z_0 = 0.$$

Example: $c = 1$:

$$0, 1, 4, 9, 16 \ldots.$$

Example: $c = -1$:

$$0, -1, 0, -1, 0, \ldots.$$

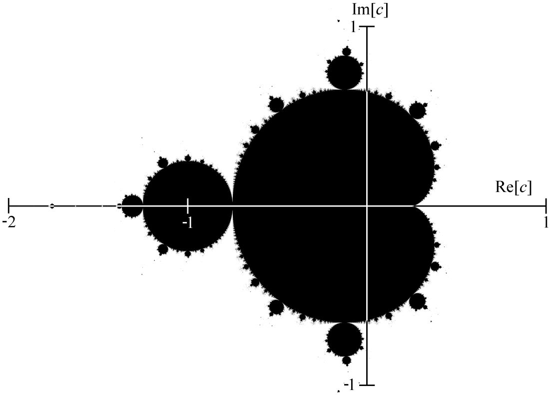


Mandelbrot Set - $z_{n+1} = z_n^2 + c, \quad z_0 = 0.$

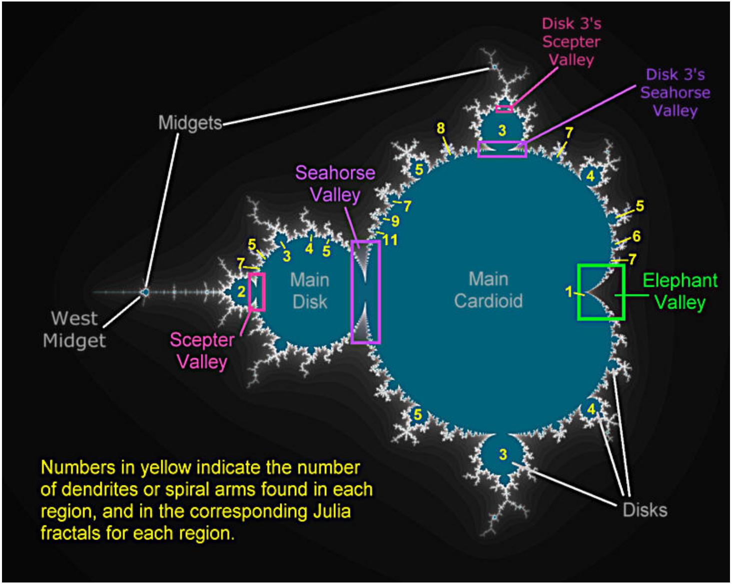
Example: $c = i$:

$$\begin{aligned} x_0 &= 0 \\ x_1 &= 0^2 + i = i \\ x_2 &= i^2 + i = -1 + i \\ x_3 &= (-1+i)^2 + i = -i \\ x_4 &= (-i)^2 + i = -1 + i \end{aligned}$$

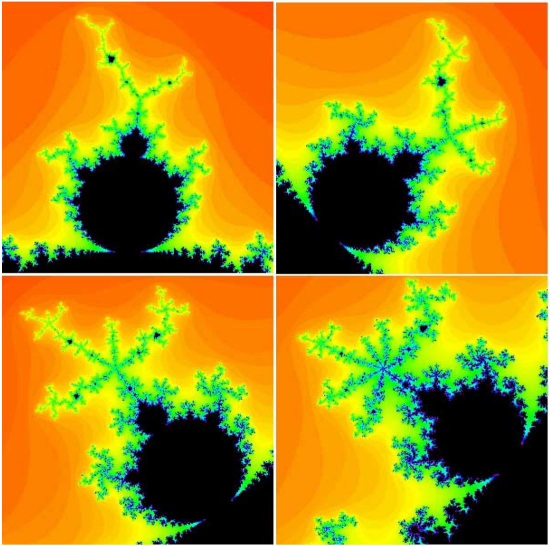
Gives period 2 orbit.



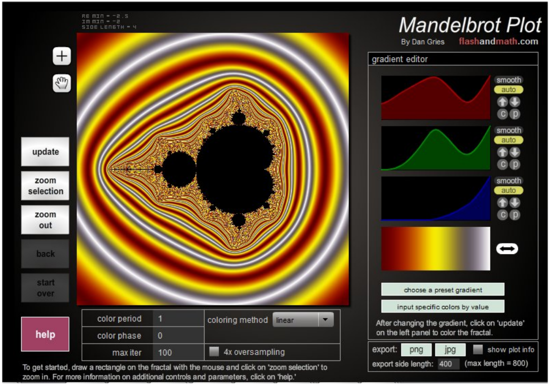
Mandelbrot Set - Bulbs



Mandelbrot Set - Bulbs



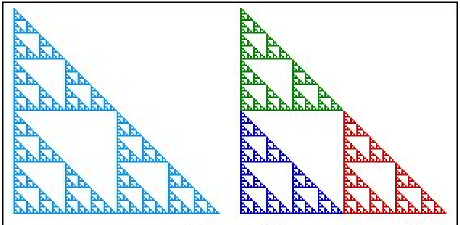
Mandelbrot Set - Plot and Zoom



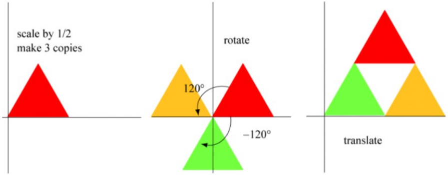
http://www.flashandmath.com/advanced/mandelbrot/MandelbrotPlot.html

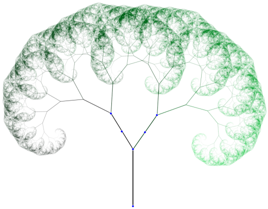
Ferns - Example of Self-Similaity

- Iterated Function Systems
- Scalings and Translations

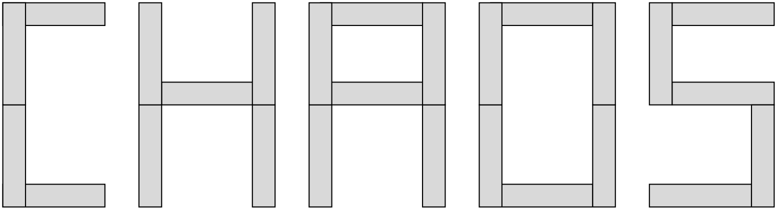


r	s	θ	φ	e	f
0.5	0.5	0	0	0.0	0.0
0.5	0.5	0	0	0.5	0.0
0.5	0.5	0	0	0.0	0.5

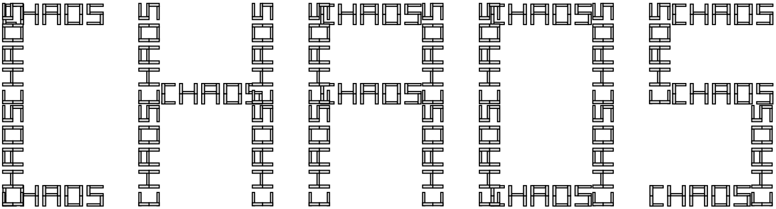




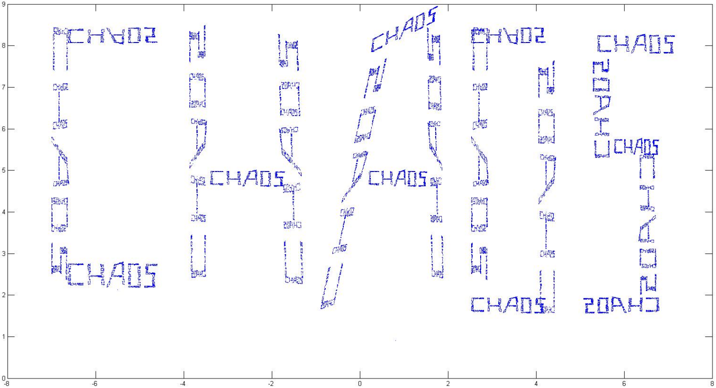
IFS - Turning CHAOS into a Fractal



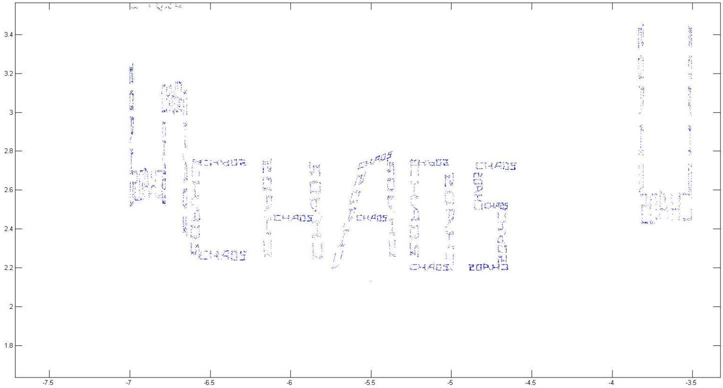
IFS - CHAOS



IFS Chaos

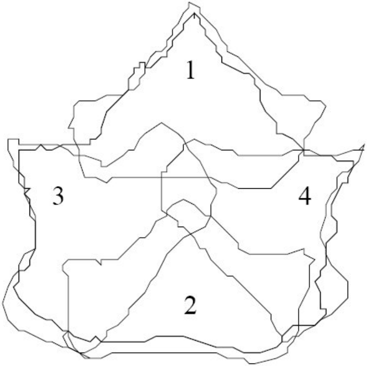


IFS Chaos



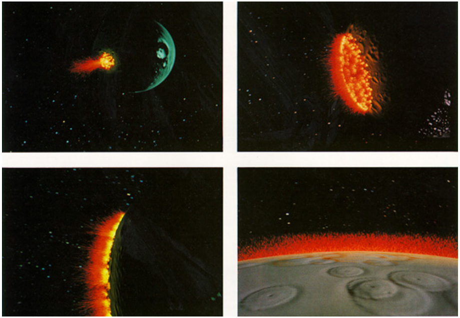
Maple Leaf - Barnsley *Fractals Everywhere*

The Collage Theorem and Fractal Image Compression.



The Genesis Effect - *Star Trek II: The Wrath of Khan* (1982)

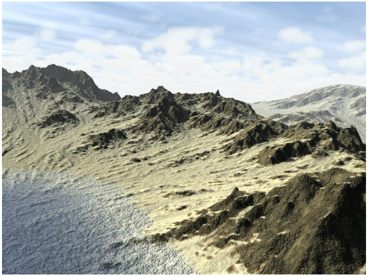
- Fractal Landscapes
- First completely computer-generated sequence in a film `http://design.osu.edu/carlson/history/tree/images/pages/genesis1_jpeg.htm`

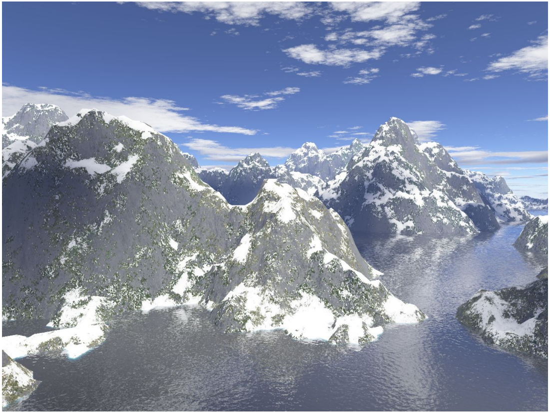


`https://www.youtube.com/watch?v=QXbWCrzWJo4`

Fractal Landscapes - Roughness in Nature

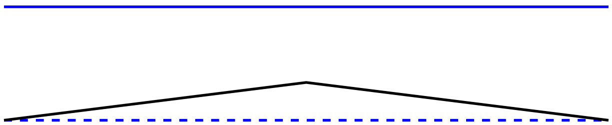
- Mountains
- Clouds



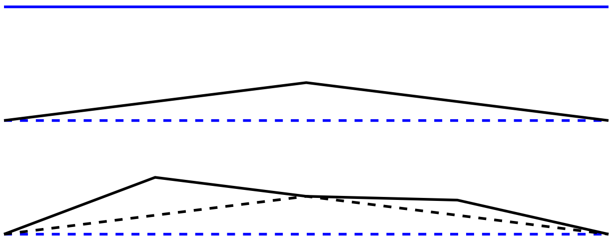


1D Midpoint Displacement Algorithm

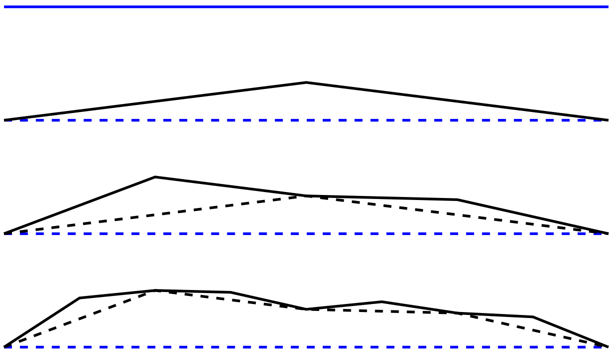
1D Midpoint Displacement Algorithm



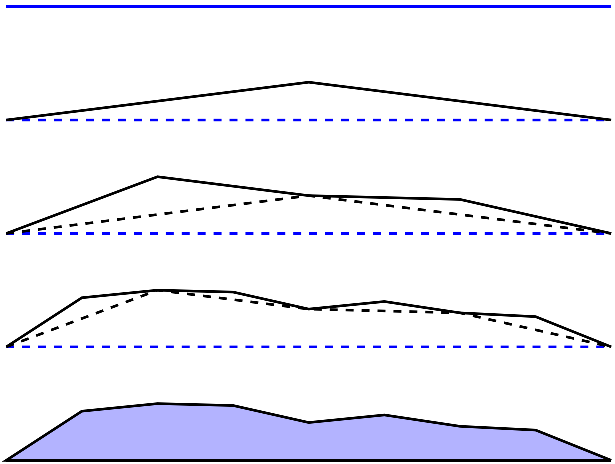
1D Midpoint Displacement Algorithm



1D Midpoint Displacement Algorithm

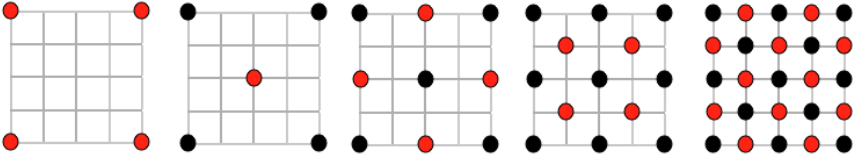


1D Midpoint Displacement Algorithm



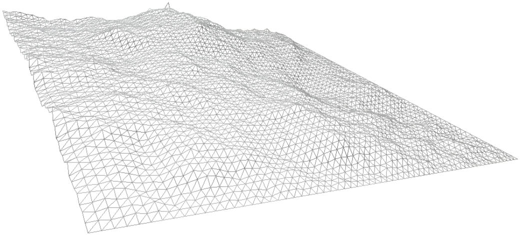
Diamond - Square Algorithm

- Square of size $2^n + 1$.
- Find Midpoint, adding random small hieghts.
- Create Diamond.
- Edge midpoints, . . .

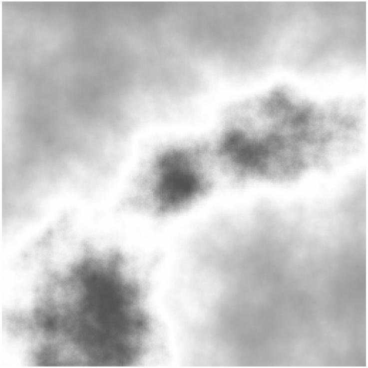


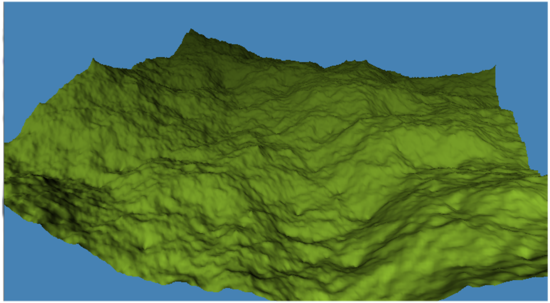
Diamond - Square Algorithm

- Square of size $2^n + 1$.
- Find Midpoint, adding random small hieghts.
- Create Diamond.
- Edge midpoints, ...



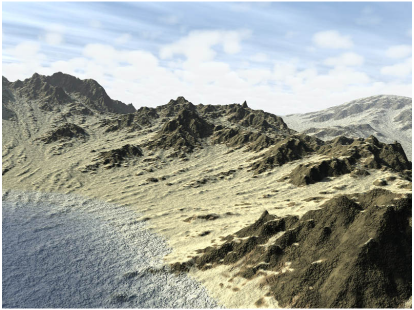
Height Maps: Clouds and Coloring





Conclusion

- Chaos - Sensitivity to initial conditions
- Fractals
- Used in movies and video games



- J. Gleick, *Chaos, Making a New Science*, 1987/2008
- E. Lorenz, *The Essence of Chaos, Making a New Science*, 1995
- B. Mandelbrot, *The Fractal Geometry of Nature*, 1982
- Barnsley, *Fractals Everywhere*, 1988/2012