Approximate Solutions of Hard Problems

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Outline

Approximating π

- $\frac{22}{7} = 3.142857\ldots.$
- $\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right).$

Wikipedia

Note that $\tan^{-1} 1 = \frac{\pi}{4}$.
Consider

$$\tan^{-1}\epsilon = \epsilon - \frac{1}{3}\epsilon^3 + \frac{1}{5}\epsilon^5 - \cdots, \quad |\epsilon| < 1.$$

or

$$\tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \cdots.$$

This is slowly converging.

Convergent series - Fix ϵ, accuracy with more terms
Asymptotic series - Fix number of terms - accuracy as $\epsilon \to 0$.

Assuming nth term

$$a_n = a + \alpha q^n, \quad |q| < 1.$$

So, $a_n \to a$ as $n \to \infty$. Can one find a from a_{n+1}, a_n, a_{n-1}?

The "answer" found from

$$S(a_n) = \frac{a_{n+1}a_{n-1} - a_n^2}{a_{n+1} + a_{n-1} - 2a_n}.$$

a_n	$S(a_n)$	Error	$S^2(a_n)$	Error	$S^3(a_n)$	Error
4.0000						
2.6667	3.166667	2.51×10^{-2}				
3.4667	3.133333	-8.26×10^{-3}	3.142105	5.13×10^{-4}		
2.8952	3.145238	3.65×10^{-3}	3.141450	-1.42×10^{-4}	3.141599	6.70×10^{-6}
3.3397	3.139682	-1.91×10^{-3}	3.141643	5.07×10^{-5}	3.141591	-1.79×10^{-6}
2.9760	3.142713	1.12×10^{-3}	3.141571	-2.14×10^{-5}	3.141593	5.77×10^{-7}
3.2373	3.140881	-7.11×10^{-4}	3.141603	1.02×10^{-5}	3.141592	-2.15×10^{-7}

Asymptotic Series Example

Example

Evaluate for small ϵ:

$$\int_0^\infty \frac{e^{-t/\epsilon}}{1+t}\,dt$$

Main contribution when t small.

$$\begin{aligned}\int_0^\infty \frac{e^{-t/\epsilon}}{1+t}\,dt &= \int_0^\infty e^{-t/\epsilon}\sum_{n=0}^\infty t^n\,dt \\ &\sim \epsilon - 1!\epsilon^2 + 2!\epsilon^3 - 3!\epsilon^4 + \cdots \qquad (1)\end{aligned}$$

The series is divergent! It is also asymptotic. The error is

$$\begin{aligned}R_1 &= -1!\epsilon^2 + 2!\epsilon^3 - 3!\epsilon^4 + \cdots \qquad (2)\\ &= \int_0^\infty \frac{te^{-t/\epsilon}}{1+t}\,dt < \int_0^\infty te^{-t/\epsilon}\,dt \approx \epsilon^2. \qquad (3)\end{aligned}$$

Error Function for Small z

$$\boxed{\operatorname{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2}\,dt}$$

e^{-t^2} is analytic; i.e., $e^{-t^2} = \sum_{n=0}^{\infty}[(-t^2)^n/n!$. Therefore,

$$\begin{aligned}\operatorname{Erf}(z) &= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)n!} \\ &= \frac{2}{\sqrt{\pi}} \left(z - \frac{1}{3}z^3 + \frac{1}{10}z^5 - \frac{1}{42}z^7 + \cdots \right) \qquad (4)\end{aligned}$$

Accuracy of 10^{-5} :

Terms	Up to $z =$
8	1
16	2
31	3
75	5

Computer cannot give correct answer to 10^{-4} at $z = 3$ due to round-off error.

Error Function - Large z

$$\boxed{\operatorname{Erf}(z) = 1 - Erfc(z) = 1 - \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2}\, dt}$$

Integrate by parts,

$$\begin{aligned} \int_z^\infty e^{-t^2}\, dt &= \frac{e^{-z^2}}{2z} - \int_z^\infty \frac{e^{-t^2}}{2t^2}\, dt \\ &= \frac{e^{-z^2}}{2z}\left(1 - \frac{1}{2z^2} + \frac{3(1)}{(2z^2)^2} - \cdots\right). \end{aligned} \qquad (5)$$

$$\operatorname{Erfc}(z) = \frac{e^{-z^2}}{z\sqrt{\pi}}\left(1 - \frac{1}{2z^2} + \frac{3(1)}{(2z^2)^2} - \frac{5(3)(1)}{(2z^2)^3} + \mathcal{O}(z^{-8})\right)$$

At $z = 2.5$, only need three terms for accuracy of 10^{-5} and only need two terms when $z > 3$. However,

This expansion is divergent!

Plot of Erfc (z) and Leading Term

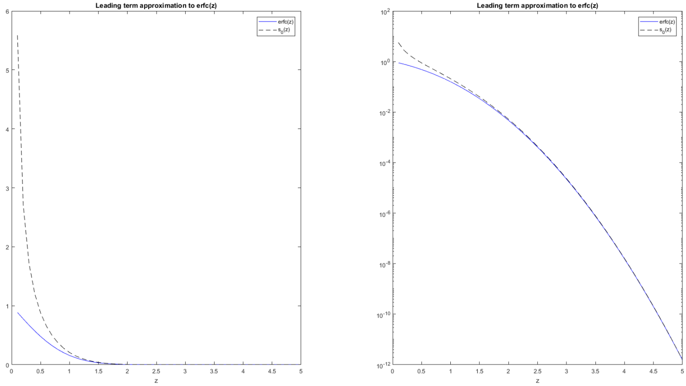
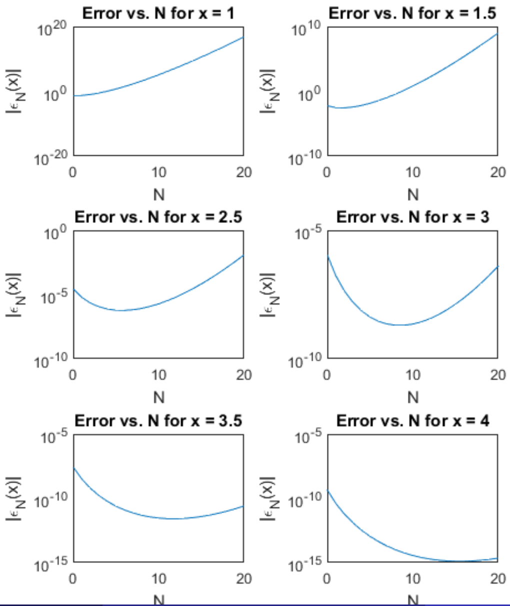


Figure 1: Caption

Plot of Errors vs N



Watson's Lemma and Integral Asypmtotics

$$I(x) = \int_0^b f(t)e^{-xt}\,dt, \quad b > 0.$$

1. $f(t)$ is continuous on $0 \leqslant t \leqslant b$,
2. $f(t) \sim t^\alpha \sum_{n=0}^\infty a_n t^{\beta n}$ as $t \to 0^+$, $\alpha > -1$, $\beta > 0$.
3. Major contribution for large x from $t = \mathcal{O}(1/x)$.

Then

$$I(x) \sim \sum_{n=0}^\infty \frac{a_n \Gamma(\alpha + \beta n + 1)}{x^{\alpha+\beta n+1}}, \quad x \to \infty$$

Methods for finding asymptotic behavior as $x \to \infty$

- Laplace's Method: $\int_a^b f(t)e^{x\phi(t)}\,dt$
- Method of Stationary Phase: $\int_a^b f(t)e^{xi\psi(t)}\,dt$
- Method of Steepest Descent: $\int_C h(t)e^{x\rho(t)}\,dt$,

$h(t)$ $\rho(t)$ are analytic, C complex contour.
$f(t)$, $\phi(t)$, $\psi(t)$ are real, continuous functions.

Example

Find the roots of $y = x^3 - 2x^2 + \epsilon x$ for small ϵ.

If $\epsilon = 0$, $y = x^3 - 2x^2$. The roots are $x = 2, 0, 0$.

Near $x = 2$: Let $x = 2 + a\epsilon + b\epsilon^2 + \cdots$. Then,

$$\begin{aligned} y &= (2 + a\epsilon + b\epsilon^2 + \cdots)^3 - 2(2 + a\epsilon + b\epsilon^2 + \cdots)^2 \\ &\quad + \epsilon(2 + a\epsilon + b\epsilon^2 + \cdots). \\ &= (4a + 2)\epsilon + (4b + 4a^2 + a)\epsilon^2 + \cdots . \end{aligned} \tag{6}$$

Set $y = 0$. Then, $a = -\frac{1}{2}$, $x \approx 2 - \frac{\epsilon}{2}$.

Plot of $y = x^3 - 2x^2 + \epsilon x$, $\epsilon = 0.1$

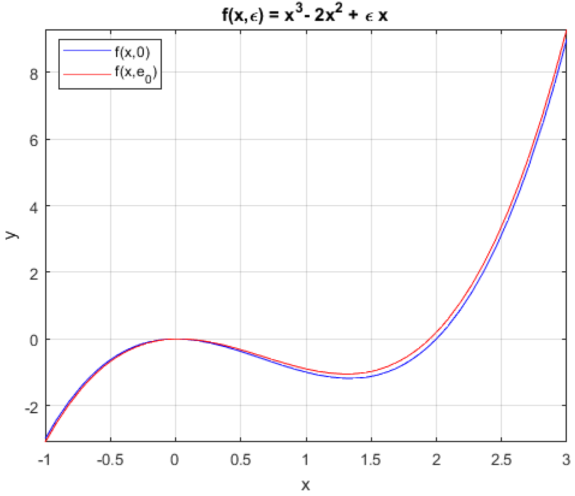


Figure 3: Caption

Magnified Plot of $y = x^3 - 2x^2 + \epsilon x$, $\epsilon = 0.1$

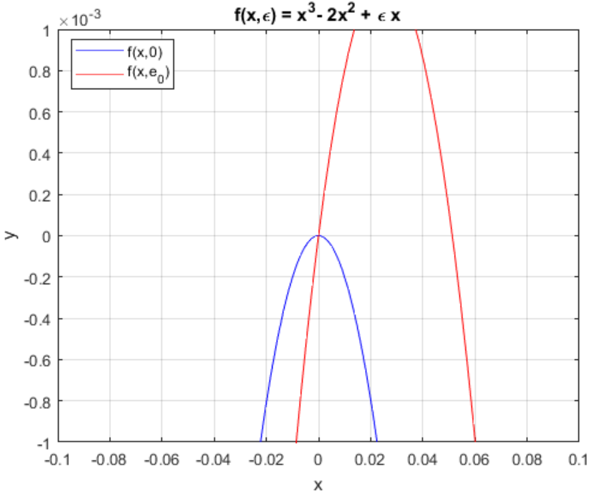


Figure 4: Caption

Roots near $x = 0$ for $y = x^3 - 2x^2 + \epsilon x$

The dominant terms are x^2 and ϵx. Then, if $x \sim \epsilon$, we have $y \sim \epsilon^2$.

Rescale variables: $x = \epsilon X$ and $y = \epsilon^2 Y$. Then,

$$Y = \epsilon X^3 - 2X^2 + X.$$

For $\epsilon = 0$, roots are $X = 0, \frac{1}{2}$, or $x = 0, \frac{1}{2}\epsilon + \ldots$.

To improve approximation, set $X = \frac{1}{2} + a\epsilon + b\epsilon^2 + \ldots$.

As before, find $a = \frac{1}{8}$. Then, $x = \frac{1}{2}\epsilon + \frac{1}{8}\epsilon^2 + \ldots$.

Plot of $Y = \epsilon X^3 - 2X^2 + X$, $\epsilon = 0.1$

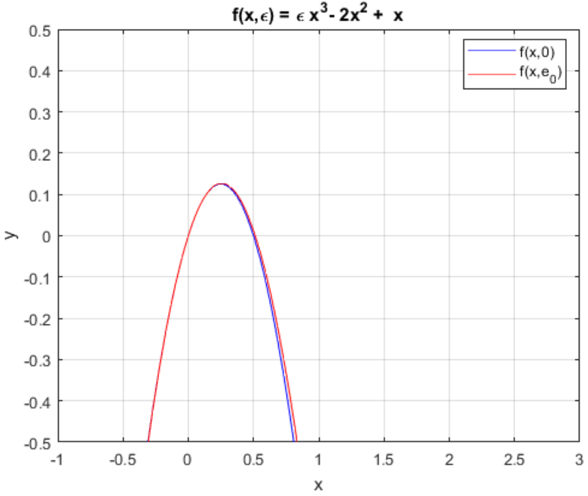


Figure 5: Caption

Rescaled Plot of $Y = \epsilon X^3 - 2X^2 + X$, $\epsilon = 0.1$

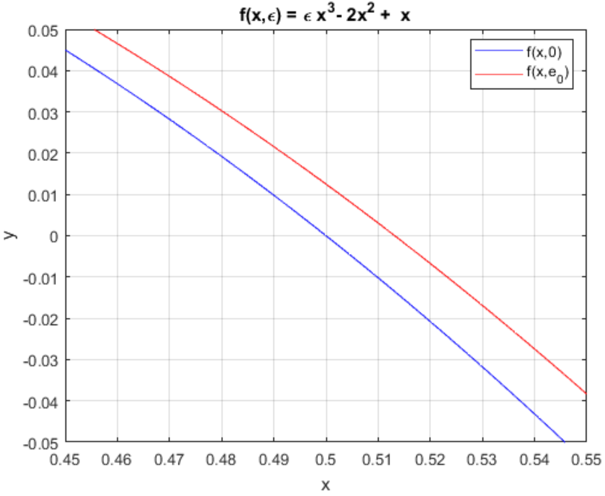


Figure 6: Caption

Initial Value Problem: $y' - \epsilon y = x, \quad y(0) = 1,$

From the exact solution get an ϵ expansion:

$$\begin{aligned} y(x;\epsilon) &= \left(1+\frac{1}{\epsilon^2}\right)e^{\epsilon x} - \frac{x}{\epsilon} - \frac{1}{\epsilon^2}. && (7) \\ &\approx \left(1+\frac{1}{\epsilon^2}\right)\left(1+\epsilon x + \frac{1}{2}\epsilon^2 x^2 + \frac{1}{6}\epsilon^3 x^3\right) - \frac{x}{\epsilon} - \frac{1}{\epsilon^2} \\ &= 1 + \epsilon x + \frac{1}{2}x^2 + \frac{1}{6}\epsilon x^3. && (8) \end{aligned}$$

On the other hand, we assume $y(x;\epsilon) = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots$. Then

$$y_0' + \epsilon y_1' + \epsilon^2 y_2' - \epsilon y_0 - \epsilon^2 y_1 + \cdots = x, \quad y_1(0) + \epsilon y_1(0) + \cdots = 1.$$

Zeroth Order: $y_0' = x$, $y_0(0) = 1.$ $\Rightarrow y_0(x) = \frac{1}{2}x^2 + 1.$
First Order: $y_1' = y_0$, $y_1(0) = 0.$ $\Rightarrow y_0(x) = x + \frac{1}{6}x^3.$

Then, $y(x) = \frac{1}{2}x^2 + 1 + \epsilon\left(x + \frac{1}{6}x^3\right) = 1 + \epsilon x + \frac{1}{2}x^2 + \frac{1}{6}\epsilon x^3.$

Boundary Value Problem
$\epsilon y'' + 2y' + 2y = 0, \quad y(0) = 0, \quad y(1) = 1.$

Exact solution:

$$y(x) = \frac{e^{m_1 x} - e^{m_2 x}}{e^{m_1} - e^{m_2}}, \text{ where } m_{1,2} = \frac{-1 \pm \sqrt{1 - 2\epsilon}}{\epsilon}.$$

Zeroth order, $\epsilon = 0$.

$$y_0(x) = ee^{-x}$$

Rescale $x = \epsilon X$, $y(x) = Y(X)$. [Balance first two terms]

$$Y'' + 2Y' + 2\epsilon Y = 0,$$

Solve $Y_0'' + 2Y_0' = 0, \quad Y_0(0) = 0.$

$$Y_0 = A + Be^{-2X} = A(1 - e^{-2X}).$$

Find A.

Matched Asymptotic Expansions

BVP:

$$\epsilon y'' + 2y' + 2y = 0, \quad y(0) = 0, \quad y(1) = 1.$$

Two Approximate Solutions: $(x = \epsilon X)$

$$\begin{aligned} y_0(x) &= ee^{-x}, \quad x \ll 1. & (9) \\ Y_0(X) &= A(1 - e^{-2X}), \quad x \gg \epsilon. & (10) \end{aligned}$$

Match on overlap $\epsilon \ll x \ll 1$. As $x \to 0$, $X \to \infty$. Thus, $A = e$.

Matched Expansion $\epsilon = 0.2$

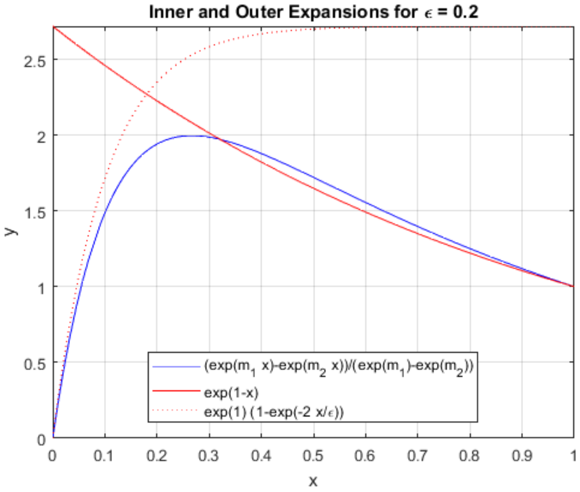


Figure 7: Caption

Matched Expansion $\epsilon = 0.1$

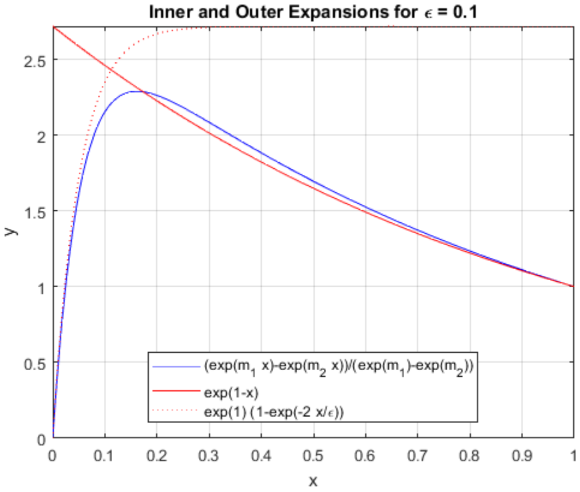


Figure 8: Caption

Perturbed Parital Differential Equation

A Perturbed mKdV Equation from M. Saravanan

$$u_\tau + \frac{3}{2}(A - \alpha f)u^2 u_\zeta + (A - \alpha f)u_{\zeta\zeta\zeta} = \alpha P, \tag{11}$$

$$P = f_{\zeta\zeta}u_\zeta + 3f_\zeta u_{\zeta\zeta} - f_{\zeta\zeta\zeta}u + f_\zeta u^3 + \frac{1}{2}u_\zeta \int_{-\infty}^{\zeta} f_{\zeta'} u^2 \, d\zeta'. \tag{12}$$

Setting $\alpha = 0$, one obtains

$$u_\tau + \frac{3}{2}Au^2 u_\zeta + Au_{\zeta\zeta\zeta} = 0. \tag{13}$$

Letting $t = A\tau, x = \zeta, u = 2v$, we obtain the mKdV equation,

$$\boxed{v_t + 6v^2 v_x + v_{xxx} = 0} \tag{14}$$

Modified KdV Equation, $v_t + 6v^2 v_x + v_{xxx} = 0$

A solution of the mKdV Equation is the one soliton solution,

$$v(x,t) = \eta \operatorname{sech} \eta(x - \eta^2 t - x_0).$$

It has amplitude η and speed η^2.

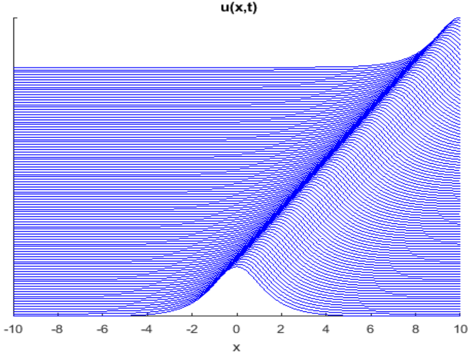


Figure 9: The unperturbed solution with $\eta = 1$ for $t \in [0, 5]$.

Perturbed mKdV Equation, $v_t + 6v^2 v_x + v_{xxx} = \epsilon F[u]$.

We seek a solution which is close to the mKdV soliton solution,

$$v_0(z) = \eta \operatorname{sech} z, \quad \text{where} \quad z = \eta(x - \xi) \quad \text{and} \quad \xi_t = \eta^2.$$

First, we introduce **multiple time scales**,

$$\partial_t = \partial_{t_1} + \epsilon \partial_{t_2}. \tag{15}$$

Then, we **expand** $v(x, t)$ as

$$v(x,t) = v_0(x, t_1) + \epsilon v_1(x, t_1, t_2) + \ldots, \tag{16}$$

assuming that $\eta = \eta(t_2)$ and $\xi = \xi(t_2)$.
Insert these expansions into the perturbed mKdV equation,

$$\boxed{\begin{aligned} v_{0t_1} + 6{v_0}^2 v_{0x} + v_{0xxx} &= 0, & (17)\\ v_{1t_1} + (6{v_0}^2 v_1)_x + v_{1xxx} &= F[v_0] - v_{0t_2}. & (18) \end{aligned}}$$

First Order Solution

$v_1(x, t_1, t_2)$ satisfies a **linearized mKdV equation**:

$$\boxed{v_{1t_1} + (6{v_0}^2 v_1)_x + v_{1xxx} = F[v_0] - v_{0t_2}}$$

Since $v_0(x, t_1) = \eta \operatorname{sech} z$ with $z = \eta(x - \xi)$ and $\xi_{t_1} = \eta^2$,

$$\begin{aligned} v_{0t_2} &= \eta_{t_2} \operatorname{sech} z - \eta \operatorname{sech} z \tanh z \frac{dz}{dt_2} \\ &= \eta_{t_2} (\operatorname{sech} z - z \operatorname{sech} z \tanh z) + \eta^2 \xi_{t_2} \operatorname{sech} z \tanh z \\ &\equiv \eta_{t_2} \phi_1(z) + \eta^2 \xi_{t_2} \phi_2(z). \end{aligned} \tag{19}$$

Then, we have

$$\boxed{v_{1t_1} + \eta^3 \hat{L} v_1 = F[v_0] - \eta_{t_2} \phi_1(z) - \eta^2 \xi_{t_2} \phi_2(z) \equiv \mathcal{F}(z), \quad (20)}$$

where

$$\hat{L} = \frac{d^3}{dz^3} + \frac{d}{dz}(6 \operatorname{sech}^2 z - 1).$$

Eigenvalue Problem for $\hat{L}$

We **seek the eigenfunctions** $\phi(z,k)$ and $\psi(z,k)$ satisfying the eigenvalue problems

$$\begin{aligned}\hat{L}\phi &= \lambda\phi, \\ \hat{L}^\dagger\psi &= \lambda'\psi, \end{aligned} \tag{21}$$

where

$$\begin{aligned}\hat{L} &= \frac{d^3}{dz^3} + \frac{d}{dz}(6\,\text{sech}^2 z - 1) \\ &= \frac{d^3}{dz^3} + (6\,\text{sech}^2 z - 1)\frac{d}{dz} - 12\,\text{sech}^2 z \tanh z. \end{aligned} \tag{22}$$

$$\hat{L}^\dagger = \frac{d^3}{dz^3} + (6\,\text{sech}^2 z - 1)\frac{d}{dz}. \tag{23}$$

Note the definition of an **adjoint operator**:

$$\int_{-\infty}^{\infty} u\hat{L}v\,dz = \int_{-\infty}^{\infty} (\hat{L}^\dagger u)v\,dz.$$

Solution of the Eigenvalue Problem for $\hat{L}$

Goal: Express v_1 as a **perturbation expansion** over the eigenstates of $\hat{L}$,

$$v_1(z,t) = \int_{-\infty}^{\infty} U(t,k)\phi(z,k)\,dk + \sum_{j=1}^{2} U_j(t)\phi_j(z). \qquad (24)$$

Require **orthogonality conditions**

$$\begin{aligned}
\int_{-\infty}^{\infty} \phi(z,k)\bar{\psi}(z,k')\,dz &= \delta(k-k'), \\
\int_{-\infty}^{\infty} \phi_j(z)\bar{\psi}_\ell(z)\,dz &= \delta_{j,\ell}, \qquad j,k = 1,2,
\end{aligned} \qquad (25)$$

and a **completeness relation**,

$$P\int_{-\infty}^{\infty} \phi(z,k)\bar{\psi}(z',k)\,dk + \sum_{j=1}^{2} \phi_j(z)\bar{\psi}_j(z') = \delta(z-z').$$

Relation to Solution

Recall that the first order correction satisfies the equation

$$v_{1\,t_1} + \eta^3 \hat{L} v_1 = \mathcal{F}(z), \tag{26}$$

where $\mathcal{F}(z) = F[v_0] - \eta_{t_2}\phi_1(z) - \eta^2 \xi_{t_2}\phi_2(z)$.
We assume

$$v_1(z, t_1) = \int_{-\infty}^{\infty} U(t_1, k)\phi(z, k)\, dk + \sum_{j=1}^{2} U_j(t_1)\phi_j(z). \tag{27}$$

Inserting this form into Equation (26), we have

$$\begin{aligned} \mathcal{F}(z) &= \int_{-\infty}^{\infty} \left[U_{t_1}(t_1, k) + \lambda\eta^3 U(t_1, k)\right] \phi(z, k)\, dk \\ &\quad + U_1'(t_1)\phi_1(z) + \left[U_2'(t_1) - 2\eta^3 U_1(t_1)\right] \phi_2(z). \end{aligned} \tag{28}$$

Expanding $\mathcal{F}(z)$ in the basis, we can solve for the coefficients.

Solving for Coefficients

Expanding $\mathcal{F}(z)$ in the basis of eigenfunctions,

$$\mathcal{F}(z) = \int_{-\infty}^{\infty} \hat{f}(t_1, k)\phi(z, k)\, dk + \sum_{j=1}^{2} f_j(t_1)\phi_j(z), \tag{29}$$

we have the set of equations

$$\begin{aligned} U_{t_1}(t_1, k) - ik(1 + k^2)\eta^3 U(t_1, k) &= \hat{f}(t_1, k), \\ U_1'(t_1) &= f_1(t_1), \\ U_2'(t_1) - 2\eta^3 U_1(t_1) &= f_2(t_1). \end{aligned} \tag{30}$$

Solve subject to $U(0, k) = 0$, $U_1(0) = 0$, $U_2(0) = 0$. Then,

$$\hat{f}(t_1, k) = \int_{-\infty}^{\infty} \mathcal{F}(z)\bar{\psi}(z, k)\, dz \equiv \langle \mathcal{F}, \bar{\psi} \rangle, \tag{31}$$

$$f_j(t_1) = \int_{-\infty}^{\infty} \mathcal{F}(z)\psi_j(z)\, dz \equiv \langle \mathcal{F}, \psi_j \rangle, \quad j = 1, 2. \tag{32}$$

Solution of the Eigenvalue Problem for $\hat{L}$

Eigenvalues: Assuming $\psi(z,k)$ and $\phi(z,k)$ approach e^{ikz} as $|z| \to \pm\infty$,

$$\lambda = \lambda' = (ik)^3 - ik = -ik(k^2+1). \qquad (33)$$

The **continuous eigenfunctions** are given by

$$\begin{aligned} \phi(z,k) &= C\left(-1-k^2-2ik\tanh z + 2\tanh^2 z\right)e^{ikz}, & (34)\\ \psi(z,k) &= C'\left(1-k^2-2ik\tanh z\right)e^{ikz}, & (35) \end{aligned}$$

where $2\pi(1+k^2)^2 C\bar{C}' = 1$ and $\bar{\psi}(z,k) = \psi(z,-k)$.

The **discrete states** are

$$\begin{aligned} \psi_1(z) &= \operatorname{sech} z, & \psi_2(z) &= z \operatorname{sech} z, & (36)\\ \phi_1(z) &= (1 - z\tanh z)\operatorname{sech} z, & \phi_2(z) &= \operatorname{sech} z \tanh z. & (37) \end{aligned}$$

$$\begin{aligned} \hat{L}\phi_1 &= -2\phi_2, & \hat{L}\phi_2 &= 0,\\ \hat{L}^\dagger\psi_1 &= 0, & \hat{L}^\dagger\psi_2 &= 2\psi_1. & (38) \end{aligned}$$

Example: $f(\zeta) = \operatorname{sech} z$, $z = \eta(\zeta - \xi)$.

Modified KdV of M. Saravanan

We want to solve

$$u_{1_{t_1}} + \eta^3 A\hat{L}u_1 = F_1(z) + P[u_0]$$

where

$$\begin{aligned} F_1(z) &= -2\eta_{t_2}\phi_1(z) - 2\eta^2\xi_{t_2}\phi_2(z) - 2\eta^4 \operatorname{sech}^2 z \tanh z & (39)\\ P[u_0] &= 6\eta^3 f_\zeta \operatorname{sech} z - 2\eta^2 f_{\zeta\zeta} \operatorname{sech} z \tanh z - 2\eta f_{\zeta\zeta\zeta} \operatorname{sech} z \\ &\quad -4\eta^3 f_\zeta \operatorname{sech}^3 z - 4\eta^4 \operatorname{sech} z \tanh z \int_{-\infty}^{\zeta} f_{\zeta'} \operatorname{sech}^2 z' \, d\zeta' \\ P[u_0] &= -6\eta^4 \operatorname{sech}^2 z \tanh z - \frac{16}{3}\eta^4 \operatorname{sech}^4 z \tanh z. & (40) \end{aligned}$$

Example - Expansion Coefficients

The inner products with the discrete states, ψ_1 and ψ_2, computed as

$$\begin{aligned} f_1 &= \langle F_1 + P[u_0], \psi_1 \rangle = -2\eta_{t_2}. \\ f_2 &= \langle F_1 + P[u_0], \psi_2 \rangle = -2\eta^2 \xi_{t_2} - \frac{26}{15}\pi\eta^4. \end{aligned}$$

Leads to growth in time unless $f_j = 0$, so

$$\eta_{t_2} = 0, \quad \xi_{t_2} = -\frac{13}{15}\pi\eta^2. \tag{41}$$

Continuous inner products give $\hat{f}(k) = \langle P[u_0]\bar{\psi} \rangle$

$$\begin{aligned} &= C\int_{-\infty}^{\infty} \left(-6\eta^4 \operatorname{sech}^2 z \tanh z - \frac{16}{3}\eta^4 \operatorname{sech}^4 z \tanh z\right)\left(1 - k^2 + 2ik \tanh z\right) e^{-ikz}\, dz, \\ &= -\sqrt{\frac{\pi}{2}}\left(\frac{2k^6 + 25k^4 + 23k^2}{15(1+k^2)}\right)\frac{i\eta^4}{\sinh\frac{\pi k}{2}} \end{aligned} \tag{42}$$

The Solution

The first order correction is determined by

$$v_1(z,t_1) = \int_{-\infty}^{\infty} U(t_1,k)\phi(z,k)\,dk + \sum_{j=1}^{2} U_j(t_1)\phi_j(z), \tag{43}$$

where

$$\begin{aligned} U_{t_1}(t_1,k) - ik(1+k^2)\eta^3 U(t_1,k) &= \hat{f}(t_1,k), \\ U_1'(t_1) &= f_1(t_1) = -2\eta_{t_2}, \\ U_2'(t_1) - 2\eta^3 U_1(t_1) &= f_2(t_1) = -2\eta^2\xi_{t_2} - \frac{26}{15}\pi\eta^4. \end{aligned} \tag{44}$$

Note:

$$\begin{aligned} U_1(t_1) &= -2\eta_{t_2} t_1 \\ U_2(t_1) &= \int^{t_1}\left(2\eta^3 U_1(\tau) - 2\eta^2\xi_{t_2} - \frac{26}{15}\pi\eta^4\right)\,d\tau \\ &= -2\eta_{t_2}\eta^3 t_1^2 - \left(2\eta^2\xi_{t_2} + \frac{26}{15}\pi\eta^4\right)t_1 \end{aligned} \tag{45}$$

The Zeroth Order Solution

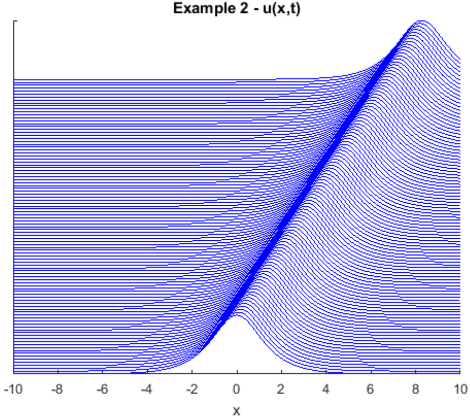


Figure 10: The zeroth order solution with $\eta = 1$ and $\alpha = 0.2$ for $t \in [0, 10]$

Location of the Zeroth Order Solution

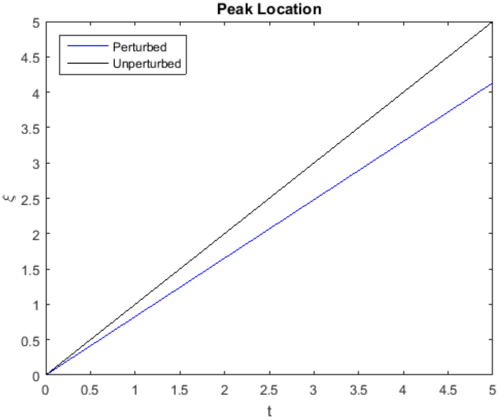


Figure 11: The location of the peak for the zeroth order solution (blue) with $\eta = 1$ and $\alpha = 0.2$ for $t \in [0, 5]$ as compared to the unperturbed solution (black).

First Order Correction, $v_1(z, t_1)$

From the inner products we have

$$v_1(z, t_1) = \int_{-\infty}^{\infty} \hat{f}(k) \frac{1 - e^{-ik(k^2+1)\eta^3 A t_1}}{-ik(k^2+1)A\eta^3} \phi(z, k)\, dk,$$

where

$$\hat{f}(k) = -\sqrt{\frac{\pi}{2}} \left(\frac{2k^6 + 25k^4 + 23k^2}{15(1+k^2)} \right) \frac{i\eta^4}{\sinh \frac{\pi k}{2}},$$

$$\phi(z, k) = \frac{1}{\sqrt{2\pi}(1+k^2)} \left(-1 - k^2 - 2ik \tanh z + 2\tanh^2 z \right) e^{ikz}.$$

Rewriting the solution, we have

$$v_1(z, t_1) = -\frac{\eta}{30A} \int_{-\infty}^{\infty} P(k, z) \frac{(1 - e^{-ik(k^2+1)\eta^3 A t_1}) e^{ikz}}{k(k^2+1)^3 \sinh \frac{\pi k}{2}}\, dk,$$

$$P(k, z) = (2k^6 + 25k^4 + 23k^2) \left(-1 - k^2 - 2ik \tanh z + 2\tanh^2 z \right).$$

Fourier Transforms of $\operatorname{sech}^p z$ and $\operatorname{sech}^p z \tanh z$

In order to compute the first order corrections, we need to compute

$$\langle P[u_0], \bar{\psi} \rangle = C' \int_{-\infty}^{\infty} P[u_0] \left(1 - k^2 + 2ik \tanh z\right) e^{-ikz}\, dz. \qquad (46)$$

This can be reduced to knowing how to compute the integrals

$$J_p(k) = \int_{-\infty}^{\infty} e^{ikz} \operatorname{sech}^p z\, dz, \qquad (47)$$

$$K_p(k) = \int_{-\infty}^{\infty} e^{ikz} \operatorname{sech}^p z \tanh z\, dz \qquad (48)$$

for p a nonnegative integer.

Using residue theory or integration by parts:

$$J_p(k) = \frac{k^2 + (p-2)^2}{(p-2)(p-1)} J_{p-2}(k) \quad p > 2, \qquad (49)$$

$$K_p(k) = \frac{ik}{p} J_p(k), \quad p > 0. \qquad (50)$$

$J_p(k)$ Recursion - Integration by Parts

$$\begin{aligned}
J_p(k) &= \int_{-\infty}^{\infty} e^{ikz} \operatorname{sech}^p z \, dz, \\
&= \int_{-\infty}^{\infty} e^{ikz} \operatorname{sech}^{p-2} z \frac{d}{dz} (\tanh z) \, dz, \quad p > 2, \\
&= \left[e^{ikz} \operatorname{sech}^{p-2} z \tanh z \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \left(e^{ikz} \operatorname{sech}^{p-2} z \right)' \tanh z \, dz, \\
&= -\int_{-\infty}^{\infty} e^{ikz} \left[ik - (p-2) \tanh z \right] \operatorname{sech}^{p-2} z \tanh z, \\
&= \frac{ik}{p-2} \left[e^{ikz} \operatorname{sech}^{p-2} z \Big|_{-\infty}^{\infty} - ik \int_{-\infty}^{\infty} e^{ikz} \operatorname{sech}^{p-2} z \, dz \right] \\
&\quad -(p-2) \int_{-\infty}^{\infty} e^{ikz} \left[\operatorname{sech}^{p-2} z - \operatorname{sech}^p z \right] dz \\
J_p(k) &= \frac{k^2 + (p-2)^2}{(p-2)(p-1)} J_{p-2}(k) \qquad (51)
\end{aligned}$$

$K_p(k)$ Recursion - Integration by Parts

The expression for $K_p(k)$ is obtained as

$$\begin{aligned}
K_p(k) &= \int_{-\infty}^{\infty} e^{ikz} \operatorname{sech}^p z \tanh z \, dz, \\
&= -\int_{-\infty}^{\infty} e^{ikz} \operatorname{sech}^{p-1} z \frac{d}{dz} \left(\operatorname{sech} z \right) \, dz, \quad p > 0, \\
&= -e^{ikz} \operatorname{sech}^p z \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \left(e^{ikz} \operatorname{sech}^{p-1} z \right)' \operatorname{sech} z \, dz, \\
&= ik \int_{-\infty}^{\infty} e^{ikz} \operatorname{sech}^p z \, dz - (p-1) \int_{-\infty}^{\infty} e^{ikz} \operatorname{sech}^p z \tanh z \, dz, \\
&= ikJ_p(k) - (p-1) K_p(k), \\
K_p(k) &= \frac{ik}{p} J_p(k), \quad p > 0. \qquad (52)
\end{aligned}$$

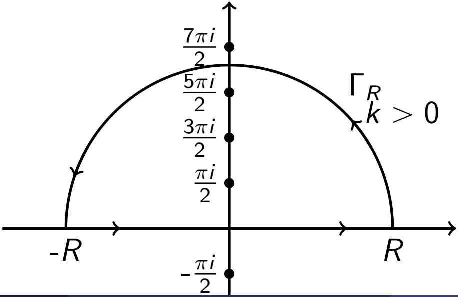
We first note

$$J_0(k) = K_0(k) = \int_{-\infty}^{\infty} e^{ikz}\, dz = 2\pi\delta(k).$$

$J_1(k)$ can be determined using contour integral methods. Writing

$$J_1(k) = \int_{-\infty}^{\infty} e^{ikz}\operatorname{sech} z\, dz = \int_{-\infty}^{\infty} \frac{e^{ikz}}{\cosh z}\, dz,$$

we see there are simple poles at $z = \frac{2n-1}{2}\pi i$ for n an integer.



Compute Residues

The Residue Theorem and Jordan's Lemma give

$$J_1(k) = 2\pi i \sum_{n=1}^{\infty} \operatorname{Res}\left[\frac{e^{ikz}}{\cosh z}; z_n = \frac{2n-1}{2}\pi i\right]$$

The residues are determined as

$$\begin{aligned}
\operatorname{Res}\left[\frac{e^{ikz}}{\cosh z}; z_n = \frac{2n-1}{2}\pi i\right] &= \lim_{z \leftarrow z_n} (z - z_n)\frac{e^{ikz}}{\cosh z} \\
&= \left.\frac{e^{ikz_n}}{\sinh z_n}\right|_{z_n = \frac{2n-1}{2}\pi i} \\
&= (-1)^n i e^{-\frac{2n-1}{2}k\pi}. \qquad (53)
\end{aligned}$$

Then,

$$\begin{aligned} J_1(k) &= 2\pi i \sum_{n=1}^{\infty} \operatorname{Res}\left[\frac{e^{ikz}}{\cosh z}; z_n = \frac{2n-1}{2}\pi i\right] \\ &= 2\pi i \sum_{n=1}^{\infty} (-1)^n i e^{-\frac{2n-1}{2}k\pi} \\ &= -2\pi e^{k\pi/2} \sum_{n=1}^{\infty} \left(-e^{-k\pi}\right)^n \\ &= 2\pi e^{k\pi/2} \frac{e^{-k\pi}}{1+e^{-k\pi}} \\ &= \frac{\pi}{\cosh \frac{\pi k}{2}}, \quad k > 0. \end{aligned} \tag{54}$$

Similarly, for

$$J_2(k) = \int_{-\infty}^{\infty} e^{ikz}\,\mathrm{sech}^2 z\,dz = \int_{-\infty}^{\infty} \frac{e^{ikz}}{\cosh^2 z}\,dz,$$

we compute the residues as

$$\begin{aligned} \mathrm{Res}\left[\frac{e^{ikz}}{\cosh^2 z}; z_n = \frac{2n-1}{2}\pi i\right] &= \lim_{z\leftarrow z_n} \frac{d}{dz}\left[(z-z_n)^2 \frac{e^{ikz}}{\cosh^2 z}\right] \\ &= -ike^{-\frac{2n-1}{2}k\pi}. \end{aligned} \tag{55}$$

Then,

$$\begin{aligned} J_2(k) &= 2\pi i \sum_{n=1}^{\infty} (-ik) e^{-\frac{2n-1}{2}k\pi} \\ &= 2\pi k e^{k\pi/2} \frac{e^{-k\pi}}{1-e^{-k\pi}} \\ &= \frac{\pi k}{\sinh \frac{\pi k}{2}}, \quad k > 0. \end{aligned} \tag{56}$$

Conclusion

1. Approximating π
2. Asymptotic Behavior of Integrals
3. Root Finding
4. Initial Value Problem
5. Boundary Value Problem
6. Perturbed Nonlinear Evolution Equation
7. Fourier Transform of $\operatorname{sech}^p z$

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