Spacetime Diagrams

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March 2, 2021

Causal Structure

There are several approaches to visualize spacetimes in general relativity. One is to look at a grid formed by the paths that light rays take. For example, consider the two-dimensional line element for a flat spacetime

$$ds^2 = -dt^2 + dx^2.$$

Here we have set $c = 1$.

Light rays travel on worldlines such that $ds^2 = 0$, Therefore,

$$\frac{dt}{dx} = \pm 1,$$

or $t = \pm x + \text{const}$. These families of solutions are shown in Figure 1. Future pointing light cones are shown where the boundaries of these light cones lie along the paths of outgoing (+1) and infalling (-1) worldlines.

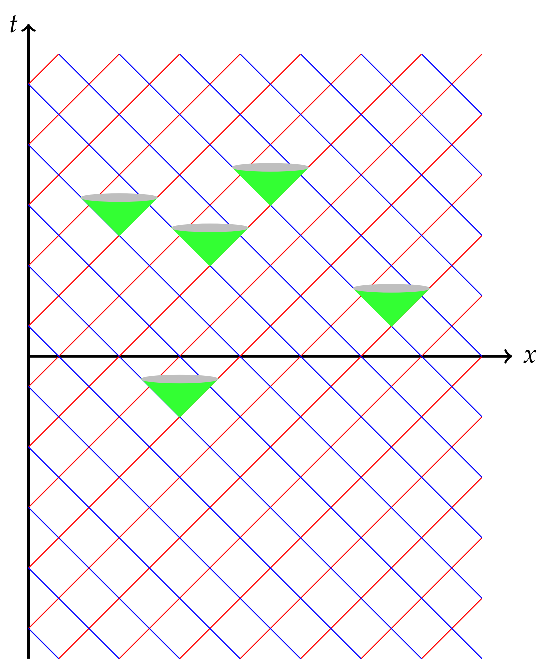

Figure 1: Plot of null curves and light-cones for $ds^2 = -dt^2 + dx^2$

Now consider the line element

$$ds^2 = -x^2 dt^2 + dx^2.$$

In order to determine light ray paths for this spacetime, we set

$$-x^2dt^2 + dx^2 = 0,$$

or

$$\frac{dx}{dt} = \pm x.$$

So, $x(t) = e^{\pm(t-t_0)}$.

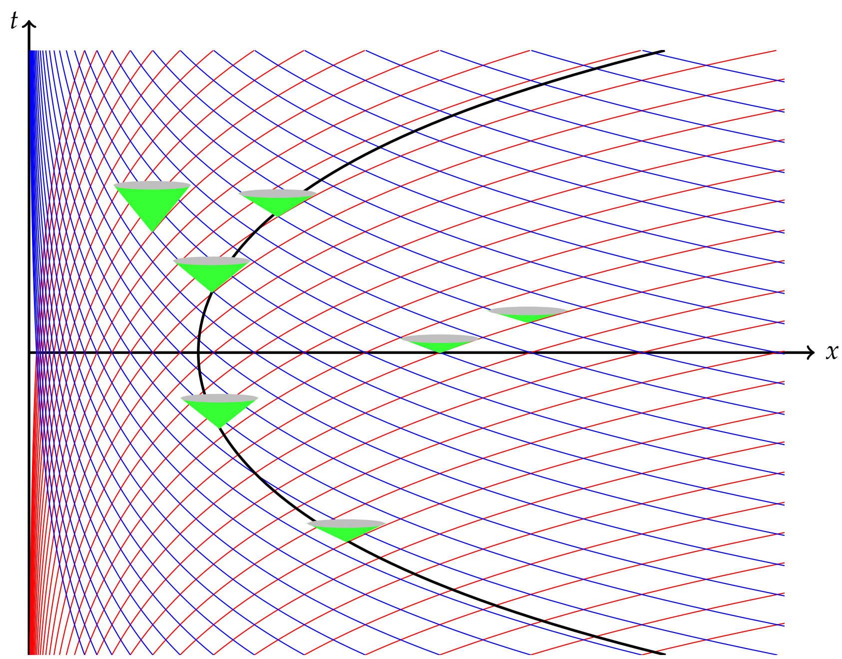

Figure 2: Plot of null curves and light-cones for $ds^2 = -x^2dt^2 + dx^2$.

In Figure 2 we plot these families of curves. Locally, we can draw future null cones whose sides are tangent to these curves at the present point. Furthermore, we drew the worldline

$$x(t) = \cosh t$$

with a few light cones on it. We see that this is a timelike path since $\frac{dx}{dt} = \sinh t < \cosh t = x$.

Schwarzschild Geometry

The line element for empty space outside a spherically symmetric source of curvature is given by the Schwarzschild line element,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{1}$$

We have used geometrized units ($c = 1$ and $G = 1$). We want to investigate the geometry by looking at its causal structure. Namely, what do the light cones look like?

We consider radial null curves. Radial null curves are curves followed by light rays ($ds^2 = 0$) for which θ and ϕ are constant. Thus,[1]

$$-\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2=0.$$

Therefore, the slope of the light cones in r-t space is given by

$$\frac{dt}{dr}=\pm\left(1-\frac{2M}{r}\right)^{-1}.$$

We note that for large r, $\frac{dt}{dr}\to\pm 1$. This indicates that for large r light rays travel as if in flat spacetime. As light rays approach $r = 2M$, $\frac{dt}{dr}\to\pm\infty$. Thus, the light cones have infinite slope and close, not allowing any causal structure. This can be seen from the solution

$$t(r)=\pm r\pm 2M\ln|r-2M|+\text{ constant.}$$

In Figure 3 we show the radial light rays for the Schwarzschild geometry. Note how the solutions on either side of $r = 2M$ suggest that no information can cross the event horizon.

This is a result of the coordinate singularity at $r = 2M$. We will explore other coordinate systems in order to see how light rays outside the event horizon, $r = 2M$, are connected to those inside.

[1] We can write this as

$$dt^2=\frac{dr^2}{1-\frac{2M}{r}}\equiv dr_*^2,$$

where r_* is called the tortoise coordinate. Integrating these equations, we have

$$r_*=r+2M\ln|r-2M|$$

and $t+r_*=$ const. We note that as r approaches $2M$, $r_*\to\infty$.

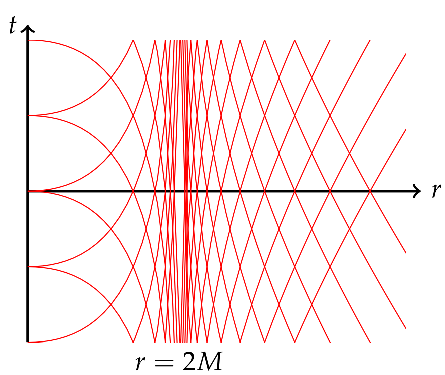

Figure 3: Radial light rays for the Schwarzschild geometry given by $t(r)=\pm r\pm 2M\ln|r-2M|+$ constant..

Eddington-Finkelstein Coordinates

We begin with the Schwarzschild line element in geometrized units ($c = 1$ and $G = 1$),

$$ds^2=-\left(1-\frac{2M}{r}\right)dt^2+\left(1-\frac{2M}{r}\right)^{-1}dr^2+r^2\left(d\theta^2+\sin^2\theta\,d\phi^2\right).$$

We will first transform this system using different sets of coordinates. The first set of coordinates are the Eddington-Finkelstein coordinates. These are defined by replacing the time variable, t, with a new time variable,

$$v = t + r_* = t + r + 2M \ln\left|\frac{r}{2M} - 1\right|.$$

The line element becomes

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right).$$

Radial light rays in this system ($ds = 0$ and $d\theta = d\phi = 0$) are found to satisfy

$$0 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr.$$

Therefore, either $dv = 0$ or

$$\begin{aligned} 0 &= -\left(1 - \frac{2M}{r}\right) dv + 2dr \\ \frac{dv}{dr} &= \frac{2r}{r - 2M} \\ &= \frac{2(r - 2M) + 4M}{r - 2M} \\ &= 2 + \frac{4M}{r - 2M} \\ v &= 2r + 4M \ln|r - 2M| + \text{const} \end{aligned}$$

Therefore, radial light rays follow lines of constant v or

$$v - 2r - 4M \ln|r - 2M| = \text{const}$$

as shown in Figure 4.

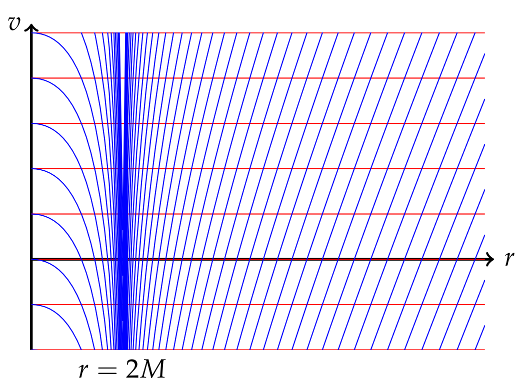

Figure 4: Radial light rays follow lines of constant v or $v = 2r + 4M \ln|r - 2M|$ in Eddington Finkelstein coordinates.

In order to have ingoing radial light rays depicted at 45°, such as we saw in Figure 1, we can instead use the new time coordinate $\tilde{t} = v - r$. Under this transformation, the radial light rays follow the paths depicted in Figure 5.

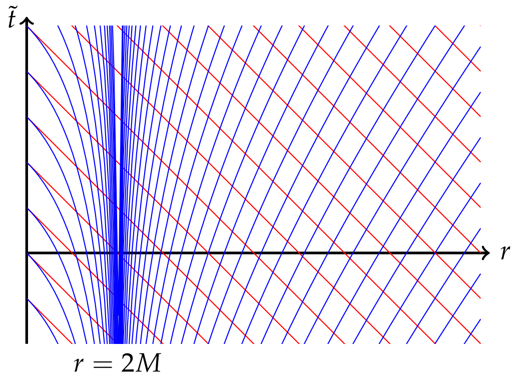

Figure 5: Radial light rays in Eddington Finkelstein coordinates $(r, \tilde{t})$, where $\tilde{t} = v - r$.

Picking a point in the spacetime, these light rays can be used to sketch the light cones. In Figure 6 we show select future null cones. As one approaches $r = 2M$, it can be seen that the light cones tip towards the event horizon. Worldlines emanating from these light cones indicate how difficult it is to bend away from the event horizon. Light cones for $r < 2M$ point towards the singularity making any massive test particle

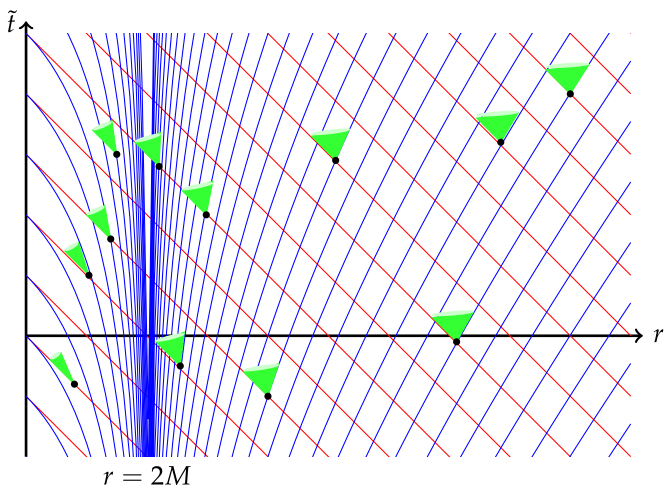

Figure 6: Radial light rays in Eddington Finkelstein coordinates $(r, \tilde{t})$ with select future light cones shown.

Kruskal-Szekeres Coordinates

We will introduce Kruskal-Szekeres coordinates for which all light rays travel at 45°. These are given by the relations

$$\begin{aligned} U^2 - V^2 &= \left(\frac{r}{2M} - 1\right) e^{r/2M} \\ \frac{V}{U} &= \tanh\left(\frac{t}{4M}\right), \quad r > 2M, \end{aligned}$$

$$\frac{U}{V} = \tanh\left(\frac{t}{4M}\right), \quad r < 2M. \tag{2}$$

These equations can be solved for the transformation mapping (t, r) into (V, U).

$$U = \begin{cases} \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \cosh \frac{t}{4M}, & r > 2M. \\ \left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} \sinh \frac{t}{4M}, & r < 2M. \end{cases} \tag{3}$$

$$V = \begin{cases} \left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} \sinh \frac{t}{4M}, & r > 2M. \\ \left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} \cosh \frac{t}{4M}, & r < 2M. \end{cases} \tag{4}$$

Under this transformation the Schwarzschild metric becomes

$$ds^2 = \frac{32M^3}{r} e^{-r/2M} \left(-dV^2 + dU^2\right) + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

for both $r > 2M$ and $r < 2M$.

In Figure 7 we show curves of constant r (blue) and constant t (red) in a Kruskal diagram. Lines of constant r are given by

$$U^2 - V^2 = \left(\frac{r}{2M} - 1\right) e^{-r/2M} = \text{const.}$$

and lines of constant t are given by

$$\frac{U}{V} = \tanh \frac{t}{4M} = \text{const for } r > 2M,$$

and

$$\frac{V}{U} = \tanh \frac{t}{4M} = \text{const for } r < 2M.$$

These give families of hyperbolae. In particular, $r = 2M$ maps to the lines $V = \pm U$ and $r = 0$ maps to lines $V^2 - U^2 = 1$. For $t = 0$, we have $V = 0$, $r > 2M$, or $U = 0$, $r < 2M$.

There are several regions of interest in Figure 7. The white region on the right is where Schwarzschild coordinates are mapped. In the region $V > -U$, the Eddington-Finkelstein coordinates are mapped.The regions III and IV in Figure 8, not depicted in Figure 7, can be interpreted as a second copy of the first region and could represent a second universe. The two universes are connected by a wormhole.

The light cones under the Kruskal-Szekeres coordinates are given by

$$ds^2 = \frac{32M^3}{r} e^{-r/2M} \left(-dV^2 + dU^2\right) = 0.$$

This gives the null rays travel along the curves $V = \pm V + \text{const}$. Therefore, the light rays travel at 45°.

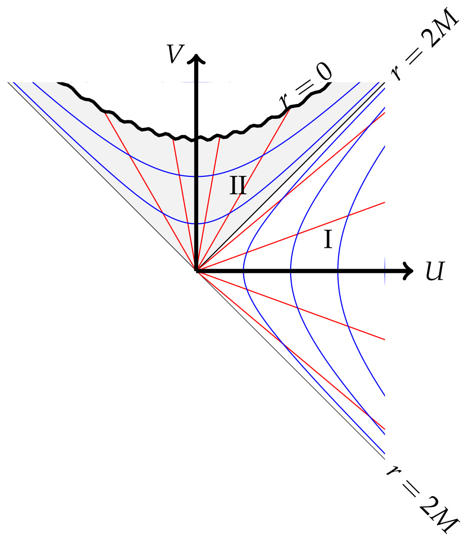

Figure 7: In this Kruskal diagram we show curves of constant r (blue) and constant t (red). The white region on the right is where Schwarzschild coordinates are mapped. In the region $V > -U$, the Eddington-Finkelstein coordinates are mapped. The singularity is at $r = 0$, which is denoted by the wavy black curve.

Penrose Diagram for Minkowski Space

The idea of a Penrose diagram is that one can introduce a transformation that maps Minkowski spacetime into a compact region so one can see the causal structure of spacetimes. We begin with the line element in spherical coordinates,

$$ds^2 = -dt^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right).$$

Now, define

$$u = t - r, \quad v = t + r.$$

Then,

$$ds^2 = -dudv + \frac{1}{4}(u-v)^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right).$$

This is a rotation[2] of the tr-axes to the uv-axes by 45°.

[2] Recall that one rotates coordinates in two dimensions using the rotation matrix

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Thus, a 45° rotation is given by

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} t \\ r \end{pmatrix} \\ &= \frac{\sqrt{2}}{2} \begin{pmatrix} t-r \\ t+r \end{pmatrix}. \end{aligned}$$

Radial light rays ($ds = 0$ and $d\theta = d\phi = 0$) give $dudv = 0$. Thus, radial light rays travel on lines of constant u and v. This is shown in Figure 9.

Now let

$$\begin{aligned} u' &= \tan^{-1} u = \frac{1}{2}(\tau - \rho), \\ v' &= \tan^{-1} v = \frac{1}{2}(\tau + \rho). \end{aligned}$$

Since $0 < r < \infty$ and $-\infty < t < \infty$, then $-\frac{\pi}{/}2 < u', v' < \pi/2$.

Rotating this primed system by 45°, we obtain a set of new coordinates,

$$\begin{aligned} \tau &= \tan^{-1} u + \tan^{-1} v = \tan^{-1}(t-r) + \tan^{-1}(t+r), \\ \rho &= \tan^{-1} v - \tan^{-1} u = \tan^{-1}(t+r) - \tan^{-1}(t-r). \end{aligned}$$

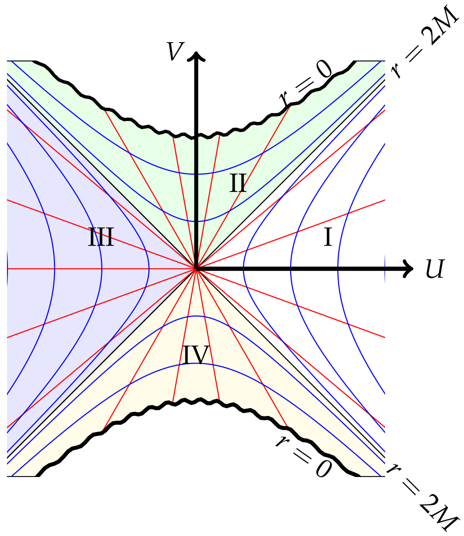

Figure 8: There are four regions of the extended Kruskal diagram. Region I is the region to which the Schwarzschild coordinates are mapped. Regions I and II are the regions to which the Kruskal-Szekeres coordinates are mapped. Regions III and IV are the extension of the Kruskal diagram, often interpreted as another, perhaps fictional, universe connected to the first by a wormhole. The second wavy black singularity next to Region IV is sometimes called a white hole since future pointing worldlines diverge from the region.

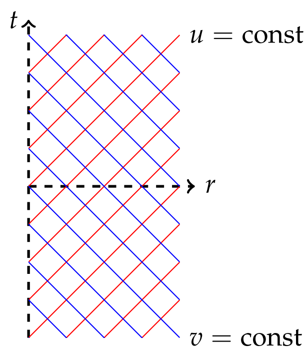

Figure 9: Radial light rays travel on lines of constant u and v.

These give the Penrose diagram for Minkowski space in Figure 10. We show on the diagram lines of constant r and t.

So, we have mapped infinity to a finite region. There are several types of infinity: I_+, I_-, I_0, and $\mathscr{I}^{\pm}$. Light rays that are outgoing follow paths $t = r + \text{const}$. They leave along lines of slope 1. They arrive at $v' = \pi/2$, or future null infinity, $\mathscr{I}^+$. This symbol is called *scri plus.* Ingoing radial lines end at *scri minus*, $\mathscr{I}^-$, past null infinity. The motion of particles start at past timelike infinity, I_- and end at future timelike infinity, I_+. Finally, spacelike trajectories arrive at spacelike infinity, I_0. These infinities are shown on the Penrose diagram in Figure 10.

Penrose Diagram from Kruskal-Szekeres Coordinates

The Kruskal-Szekeres coordinates were introduced as a way to view the causal structure of the spherically symmetric source. The next step is to construct a Penrose diagram from this system. As with the

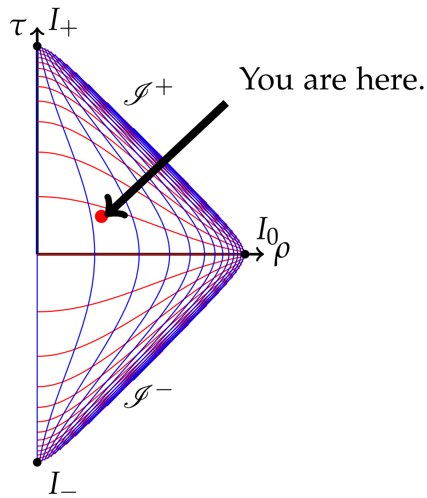

Figure 10: A Penrose diagram for Minkowski space. Lines of constant r (blue) and t (red) are shown.

Minkowski coordinates, we first rotate the coordinate system (U, V) by 45° to system (u, v) :

$$U = \frac{1}{2}(v - u), \quad V = \frac{1}{2}(v + u),$$

or

$$u = V - U, \quad v = V + U.$$

Then, similar to the construction of the Penrose diagram for flat space, we now define

$$\begin{aligned} V' &= \tan^{-1} u + \tan^{-1} v, \\ U' &= \tan^{-1} v - \tan^{-1} u. \end{aligned}$$

This will transform the infinite space to a bounded region as seen in Figure 11.

In order to look at curves of constant t or r, we want $V' = V'(t, r)$ and $U' = U'(t, r)$. Since the transformations involve the Schwarzschild radius, $r_s = 2M$, we can assume that r and t are rescaled in these units. Namely, we let $\tilde{r} = r/2M$ and $\tilde{t} = t/2M$. Then, noting that

$$V - U = \begin{cases} -\left(\frac{r}{2M} - 1\right)^{1/2} e^{(r-t)/4M}, & r > 2M. \\ \left(1 - \frac{r}{2M}\right)^{1/2} e^{(r-m)/4M}, & r < 2M. \end{cases} \tag{5}$$

$$V + U = \begin{cases} \left(\frac{r}{2M} - 1\right)^{1/2} e^{(r+t)/4M}, & r > 2M. \\ \left(1 - \frac{r}{2M}\right)^{1/2} e^{(r+t)/4M}, & r < 2M. \end{cases} \tag{6}$$

we have

$$u = \begin{cases} -(\tilde{r} - 1)^{1/2} e^{(\tilde{r} - \tilde{t})/2}, & \tilde{r} > 1. \\ (1 - \tilde{r})^{1/2} e^{(\tilde{r} - \tilde{m})/2}, & \tilde{r} < 1. \end{cases} \tag{7}$$

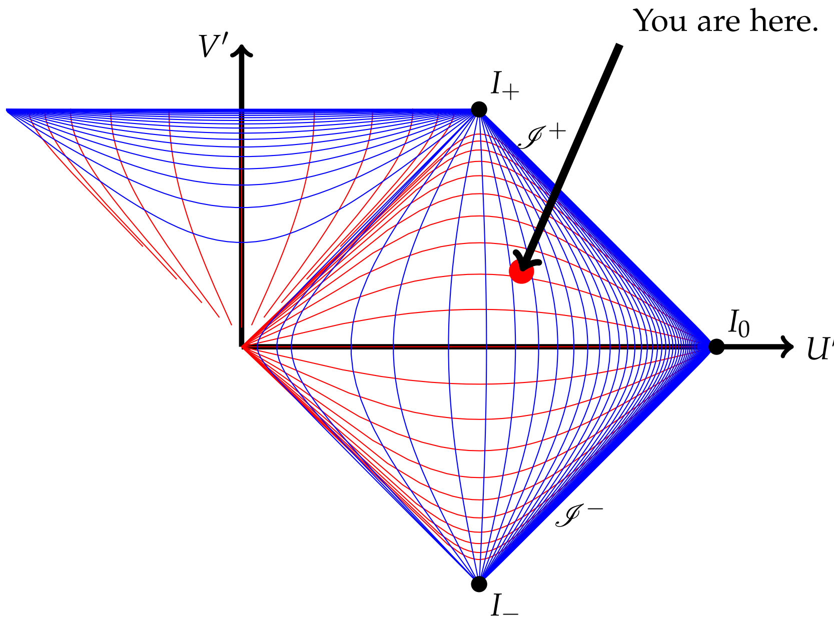

Figure 11: Penrose diagram for the Schwarzschild Geometry.

$$v = \begin{cases} (\tilde{r}-1)^{1/2}\, e^{(\tilde{r}+\tilde{t})/2}, & \tilde{r} > 1. \\ (1-\tilde{r})^{1/2}\, e^{(\tilde{r}+\tilde{m})/2}, & \tilde{r} < 1. \end{cases} \quad (8)$$

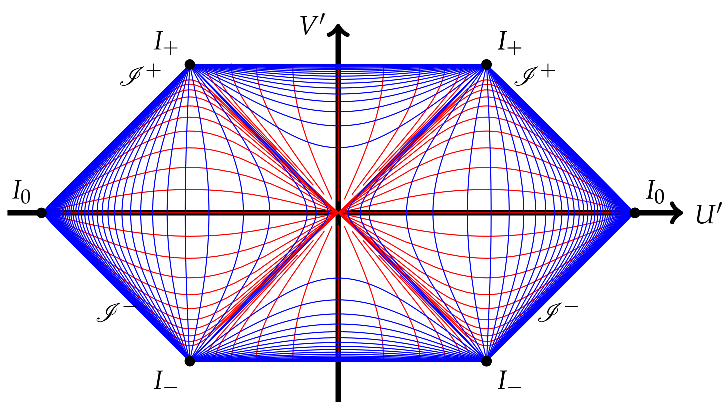

Figure 12: Penrose diagram for the maximally extended Schwarzschild solution,