

Geodesic Equations for the Wormhole Metric

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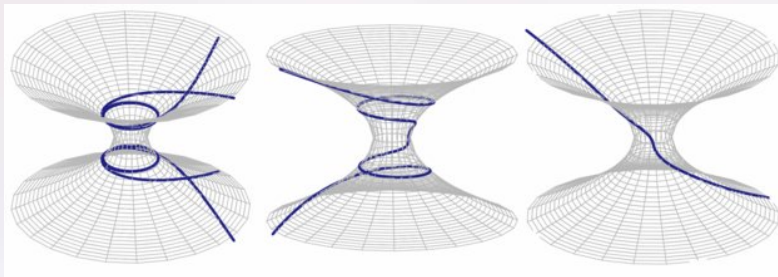
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The Wormhole Metric

Morris and Thorne wormhole metric: [M. S. Morris, K. S. Thorne, Wormholes in spacetime and their use for interstellar travel: A tool for teaching general relativity, *Am. J. Phys.* **56**, 395-412, 1988.]

$$ds^2 = -c^2 dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2 \theta d\phi^2)$$



Embedding $ds^2 = -dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2 \theta d\phi^2)$

Consider 2D slices ($t = \text{const}$, $\theta = \pi/2$). Then,

$$dS^2 = dr^2 + (b^2 + r^2) d\phi^2.$$

Compare to a cylindrical coordinate line element: $(\rho(r), \psi, z(r))$

$$\begin{aligned} d\Sigma^2 &= d\rho^2 + \rho^2 d\psi^2 + dz^2 \\ &= \left[\left(\frac{dz}{dr} \right)^2 + \left(\frac{d\rho}{dr} \right)^2 \right] dr^2 + \rho^2(r) d\phi^2. \end{aligned}$$

Then, $\rho^2 = r^2 + b^2$ and $\left(\frac{dz}{dr} \right)^2 + \left(\frac{d\rho}{dr} \right)^2 = 1$.

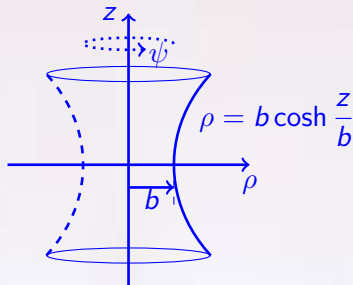
Since $\rho d\rho = r dr$, $\frac{d\rho}{dr} = \frac{r}{\rho} = \frac{r}{\sqrt{r^2 + b^2}}$. Therefore,

$$\left(\frac{dz}{dr} \right)^2 = 1 - \frac{r^2}{r^2 + b^2} = \frac{b^2}{r^2 + b^2}.$$

Embedding Diagram from $\left(\frac{dz}{dr}\right)^2 = \frac{b^2}{r^2+b^2}$

Now we integrate [substitute $r = b \sinh u$, $dr = b \cosh u du$]:

$$\begin{aligned}\frac{dz}{dr} &= \frac{b}{\sqrt{b^2 + r^2}} \\ z &= b \int \frac{dr}{\sqrt{b^2 + r^2}} \\ &= b \int \frac{b \cosh u du}{\sqrt{b^2(1 + \sinh^2 u)}}\end{aligned}$$



Therefore, $z = bu = b \sinh^{-1} \frac{r}{b}$, or

$$\rho = b \cosh \frac{z}{b}.$$

The embedded surface of revolution is a hyperboloid.

Lagrangian Approach to Geodesics

Begin with a metric $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$. Then,

$$\tau_{AB} = \int_0^1 \sqrt{-g_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma}} d\sigma.$$

Euler-Lagrange Equations \Rightarrow Geodesic Equations

$$\frac{d}{d\sigma} \left(\frac{\partial L}{\partial \dot{x}^\gamma} \right) - \frac{\partial L}{\partial x^\gamma} = 0, \quad \gamma = 0, 1, 2, 3,$$

where $\dot{x}^\gamma = \frac{dx^\gamma}{d\sigma}$ and we defined the "Lagrangian"

$$L(x^\gamma, \dot{x}^\gamma) = \sqrt{-g_{\alpha\beta} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma}} = \frac{d\tau}{d\sigma}.$$

Compute $\frac{\partial L}{\partial x^\gamma} - \frac{d}{d\sigma} \left(\frac{\partial L}{\partial(dx^\gamma/d\sigma)} \right) = 0$.

We carefully compute the derivatives for a general metric.

$$\begin{aligned}\frac{\partial L}{\partial x^\gamma} &= -\frac{1}{2L} \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} \frac{dx^\alpha}{d\sigma} \frac{dx^\beta}{d\sigma} \\ &= -\frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \cdot \\ \frac{\partial L}{\partial(dx^\gamma/d\sigma)} &= -\frac{1}{2L} g_{\alpha\beta} \left(\delta_\gamma^\alpha \frac{dx^\beta}{d\sigma} + \frac{dx^\alpha}{d\sigma} \delta_\gamma^\beta \right) \\ &= -\frac{1}{2L} \left(g_{\gamma\beta} \frac{dx^\beta}{d\sigma} + g_{\alpha\gamma} \frac{dx^\alpha}{d\sigma} \right) \\ &= -\frac{1}{L} g_{\alpha\gamma} \frac{dx^\alpha}{d\sigma}.\end{aligned}$$

The σ derivatives have been replaced by $\frac{df}{d\sigma} = \frac{df}{d\tau} \frac{d\tau}{d\sigma} = L \frac{df}{d\tau}$. We used symmetry and the fact that α and β are dummy indices.

Compute $\frac{\partial L}{\partial x^\gamma} - \frac{d}{d\sigma} \left(\frac{\partial L}{\partial(dx^\gamma/d\sigma)} \right) = 0$. (cont'd)

We differentiate the last result:

$$\begin{aligned} -\frac{d}{d\sigma} \left(\frac{\partial L}{\partial(dx^\gamma/d\sigma)} \right) &= \frac{d}{d\sigma} \left(\frac{1}{L} g_{\alpha\gamma} \frac{dx^\alpha}{d\sigma} \right) \\ &= L \frac{d}{d\tau} \left(g_{\alpha\gamma} \frac{dx^\alpha}{d\tau} \right) \\ &= L \left[g_{\alpha\gamma} \frac{d^2 x^\alpha}{d\tau^2} + \frac{dg_{\alpha\gamma}}{d\tau} \frac{dx^\alpha}{d\tau} \right] \\ &= L \left[g_{\alpha\gamma} \frac{d^2 x^\alpha}{d\tau^2} + \frac{dg_{\alpha\gamma}}{dx^\beta} \frac{dx^\beta}{d\tau} \frac{dx^\alpha}{d\tau} \right] \\ &= L \left[g_{\alpha\gamma} \frac{d^2 x^\alpha}{d\tau^2} + \frac{1}{2} \left(\frac{dg_{\alpha\gamma}}{dx^\beta} + \frac{dg_{\gamma\beta}}{dx^\alpha} \right) \frac{dx^\beta}{d\tau} \frac{dx^\alpha}{d\tau} \right]. \end{aligned}$$

We have used symmetry, re-indexing of repeated indices, and have eliminated appearances of L .

Compute $\frac{\partial L}{\partial x^\gamma} - \frac{d}{d\sigma} \left(\frac{\partial L}{\partial(dx^\gamma/d\sigma)} \right) = 0$. (finally!)

So far, we have

$$\begin{aligned} 0 &= \frac{\partial L}{\partial x^\gamma} - \frac{d}{d\sigma} \left(\frac{\partial L}{\partial(dx^\gamma/d\sigma)} \right) \\ &= L \left[g_{\alpha\gamma} \frac{d^2 x^\alpha}{d\tau^2} + \frac{1}{2} \left(\frac{dg_{\alpha\gamma}}{dx^\beta} + \frac{dg_{\gamma\beta}}{dx^\alpha} \right) \frac{dx^\beta}{d\tau} \frac{dx^\alpha}{d\tau} \right] - \frac{L}{2} \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}. \end{aligned}$$

Rearranging the terms and changing the dummy index α to δ ,

$$\begin{aligned} g_{\alpha\gamma} \frac{d^2 x^\alpha}{d\tau^2} &= \frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} - \frac{1}{2} \left(\frac{dg_{\alpha\gamma}}{dx^\beta} + \frac{dg_{\gamma\beta}}{dx^\alpha} \right) \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \\ &= -\frac{1}{2} \left[\frac{dg_{\alpha\gamma}}{dx^\beta} + \frac{dg_{\gamma\beta}}{dx^\alpha} - \frac{dg_{\alpha\beta}}{dx^\gamma} \right] \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \\ &= -\frac{1}{2} \left[\frac{dg_{\delta\gamma}}{dx^\beta} + \frac{dg_{\gamma\beta}}{dx^\delta} - \frac{dg_{\delta\beta}}{dx^\gamma} \right] \frac{dx^\delta}{d\tau} \frac{dx^\beta}{d\tau} \\ &\equiv -g_{\alpha\gamma} \Gamma_{\delta\beta}^\alpha \frac{dx^\delta}{d\tau} \frac{dx^\beta}{d\tau}. \end{aligned}$$

The Result: Key Equations

The Geodesic Equations

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma_{\beta\gamma}^\alpha \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} = 0,$$

In terms of the four-velocity:

$$\frac{du^\alpha}{d\tau} + \Gamma_{\beta\gamma}^\alpha u^\beta u^\gamma = 0.$$

The Christoffel Symbols are given by [Note: $\Gamma_{\beta\gamma}^\delta = \Gamma_{\gamma\beta}^\delta$.]

$$g_{\alpha\delta} \Gamma_{\beta\gamma}^\delta = \frac{1}{2} \left[\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} \right],$$

Wormhole Geodesics via the Lagrangian

Begin with the proper time (with $c = 1$),

$$d\tau^2 = -ds^2 = dt^2 - dr^2 - (b^2 + r^2)(d\theta^2 + \sin^2 \theta d\phi^2),$$

Write the Lagrangian,

$$L = \sqrt{\left(\frac{dt}{d\sigma}\right)^2 - \left(\frac{dr}{d\sigma}\right)^2 - (b^2 + r^2) \left(\left(\frac{d\theta}{d\sigma}\right)^2 + \sin^2 \theta \left(\frac{d\phi}{d\sigma}\right)^2 \right)},$$

Apply the Euler-Lagrange equation for each variable: t , r , θ , ϕ .

Example - time variable t , $\dot{t} \equiv \frac{dt}{d\sigma}$:

$$\frac{d}{d\sigma} \left(\frac{\partial L}{\partial \dot{t}} \right) - \frac{\partial L}{\partial t} = 0.$$

Time Equation

Lagrangian:

$$L = \left[\dot{t}^2 - \dot{r}^2 - (b^2 + r^2)(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right]^{1/2}$$

Geodesic Equation for t : [Recall that $L \frac{d}{d\tau} = \frac{d}{d\sigma}$]

$$\begin{aligned} \frac{d}{d\sigma} \left(\frac{\partial L}{\partial \dot{t}} \right) &= \frac{\partial L}{\partial t} \\ \frac{d}{d\sigma} \left(\frac{\partial L}{\partial \dot{t}} \frac{d\tau}{d\sigma} \right) &= 0 \\ L \frac{d}{d\tau} \left(\frac{d\tau}{d\tau} \right) &= 0 \end{aligned}$$

$$\boxed{\frac{d^2 t}{d\tau^2} = 0.}$$

Radial Equation

Lagrangian:

$$L = \left[\dot{t}^2 - \dot{r}^2 - (b^2 + r^2)(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right]^{1/2}$$

Geodesic Equation for r :

$$\frac{d}{d\sigma} \left(\frac{\partial L}{\partial \dot{r}} \right) = \frac{\partial L}{\partial r}$$

$$L \frac{d}{d\tau} \left(-\frac{1}{L} \frac{dr}{d\sigma} \right) = -\frac{1}{2L} (2r) \left[\left(\frac{d\theta}{d\sigma} \right)^2 + \sin^2 \theta \left(\frac{d\phi}{d\sigma} \right)^2 \right]$$

$$\frac{d^2 r}{d\tau^2} = r \left[\left(\frac{d\theta}{d\tau} \right)^2 + \sin^2 \theta \left(\frac{d\phi}{d\tau} \right)^2 \right].$$

The θ -Equation

Lagrangian:

$$L = \left[\dot{t}^2 - \dot{r}^2 - (b^2 + r^2)(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right]^{1/2}$$

Geodesic Equation for θ :

$$\frac{d}{d\sigma} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = \frac{\partial L}{\partial \theta}$$

$$L \frac{d}{d\tau} \left(-\frac{b^2 + r^2}{L} \frac{d\theta}{d\sigma} \right) = -\frac{1}{2L} (b^2 + r^2) \left[2 \sin \theta \cos \theta \left(\frac{d\phi}{d\sigma} \right)^2 \right]$$

$$\frac{d}{d\tau} \left((b^2 + r^2) \frac{d\theta}{d\tau} \right) = (b^2 + r^2) \sin \theta \cos \theta \left(\frac{d\phi}{d\tau} \right)^2.$$

The ϕ -Equation

Lagrangian:

$$L = \left[\dot{t}^2 - \dot{r}^2 - (b^2 + r^2)(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right]^{1/2}$$

Geodesic Equation for ϕ :

$$\frac{d}{d\sigma} \left(\frac{\partial L}{\partial \dot{\phi}} \right) = \frac{\partial L}{\partial \phi}$$
$$L \frac{d}{d\tau} \left(-\frac{b^2 + r^2}{L} \sin^2 \theta \frac{d\phi}{d\sigma} \right) = 0$$

$$\boxed{\frac{d}{d\tau} \left((b^2 + r^2) \sin^2 \theta \frac{d\phi}{d\tau} \right) = 0.}$$

The Geodesic Equations for the MT Wormhole

$$\frac{d^2 t}{d\tau^2} = 0$$

$$\frac{d^2 r}{d\tau^2} = r \left[\left(\frac{d\theta}{d\tau} \right)^2 + \sin^2 \theta \left(\frac{d\phi}{d\tau} \right)^2 \right]$$

$$\frac{d}{d\tau} \left((b^2 + r^2) \frac{d\theta}{d\tau} \right) = (b^2 + r^2) \sin \theta \cos \theta \left(\frac{d\phi}{d\tau} \right)^2$$

$$\frac{d}{d\tau} \left((b^2 + r^2) \sin^2 \theta \frac{d\phi}{d\tau} \right) = 0.$$

- ▶ Solve for geodesics $(t(\tau), r(\tau), \theta(\tau), \phi(\tau))$.
- ▶ Read off Christoffel Symbols, $\frac{d^2 x^\alpha}{d\tau^2} = -\Gamma_{\beta\gamma}^\alpha \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau}$

Christoffel Symbols from the Geodesic Equations

Start with general Geodesic Equation:

$$\frac{d^2 x^\alpha}{d\tau^2} = -\Gamma_{\beta\gamma}^\alpha \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau}$$

$$\frac{d^2 t}{d\tau^2} = 0.$$

$$\frac{d^2 r}{d\tau^2} = r \left[\left(\frac{d\theta}{d\tau} \right)^2 + \sin^2 \theta \left(\frac{d\phi}{d\tau} \right)^2 \right].$$

Read off the coefficients:

- ▶ $\Gamma_{\beta\gamma}^t = 0, \quad \beta, \gamma = r, \theta, \phi.$
- ▶ $\Gamma_{\theta\theta}^r = -r, \quad \Gamma_{\phi\phi}^r = -r \sin^2 \theta.$

Christoffel Symbols (cont'd)

$$\frac{d^2 x^\alpha}{d\tau^2} = -\Gamma_{\beta\gamma}^\alpha \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau}$$

$$\frac{d}{d\tau} \left((b^2 + r^2) \frac{d\theta}{d\tau} \right) = (b^2 + r^2) \sin \theta \cos \theta \left(\frac{d\phi}{d\sigma} \right)^2$$

$$(b^2 + r^2) \frac{d^2 \theta}{d\tau^2} + 2r \frac{dr}{d\tau} \frac{d\theta}{d\tau} = (b^2 + r^2) \sin \theta \cos \theta \left(\frac{d\phi}{d\sigma} \right)^2$$

$$\frac{d^2 \theta}{d\tau^2} = -\frac{2r}{b^2 + r^2} \frac{dr}{d\tau} \frac{d\theta}{d\tau} + \sin \theta \cos \theta \left(\frac{d\phi}{d\sigma} \right)^2.$$

▶ $\Gamma_{\theta r}^\theta = \frac{r}{b^2 + r^2} = \Gamma_{r\theta}^\theta$, $\Gamma_{\phi\phi}^\theta = -\sin \theta \cos \theta$.

▶ Note: $\Gamma_{\theta r}^\theta$ and $\Gamma_{r\theta}^\theta$ contribute equally, thus there is no 2.

Christoffel Symbols (cont'd)

$$\frac{d^2 x^\alpha}{d\tau^2} = -\Gamma_{\beta\gamma}^\alpha \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau}$$

$$\frac{d}{d\tau} \left((b^2 + r^2) \sin^2 \theta \frac{d\phi}{d\tau} \right) = 0.$$

$$(b^2 + r^2) \sin^2 \theta \frac{d^2 \phi}{d\tau^2} = -2r \sin^2 \theta \frac{dr}{d\tau} \frac{d\phi}{d\tau}$$

$$-2(b^2 + r^2) \sin \theta \cos \theta \frac{d\theta}{d\tau} \frac{d\phi}{d\tau}.$$

$$\frac{d^2 \phi}{d\tau^2} = -\frac{2r}{b^2 + r^2} \frac{dr}{d\tau} \frac{d\phi}{d\tau} - 2 \cot \theta \frac{d\theta}{d\tau} \frac{d\phi}{d\tau}.$$

$$\blacktriangleright \Gamma_{\phi r}^\phi = \frac{r}{b^2 + r^2} = \Gamma_{r\phi}^\phi, \quad \Gamma_{\phi\theta}^\phi = \cot \theta.$$

Christoffel Symbols from the Metric

The Christoffel symbols are defined by

$$g_{\alpha\delta}\Gamma_{\beta\gamma}^{\delta} = \frac{1}{2} \left[\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}} \right].$$

For the wormhole metric,

$$ds^2 = -dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2 \theta d\phi^2).$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b^2 + r^2 & 0 \\ 0 & 0 & 0 & (b^2 + r^2) \sin^2 \theta \end{pmatrix},$$

or, $g_{tt} = -1$, $g_{rr} = 1$, $g_{\theta\theta} = b^2 + r^2$, $g_{\phi\phi} = (b^2 + r^2) \sin^2 \theta$.

Christoffel Symbols $\Gamma_{\beta\gamma}^t$

The nonzero metric elements are

$$g_{tt} = -1, \quad g_{rr} = 1, \quad g_{\theta\theta} = b^2 + r^2, \quad g_{\phi\phi} = (b^2 + r^2) \sin^2 \theta.$$

Let $\alpha = t$ and $x^\alpha = t$, then

$$g_{t\delta} \Gamma_{\beta\gamma}^\delta = \frac{1}{2} \left[\frac{\partial g_{t\beta}}{\partial x^\gamma} + \frac{\partial g_{t\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial t} \right].$$

Since the $g_{t\mu}$ is nonzero and constant for $\mu = t$,

$$\begin{aligned} g_{tt} \Gamma_{\beta\gamma}^t &= \frac{1}{2} \left[\frac{\partial g_{t\beta}}{\partial x^\gamma} + \frac{\partial g_{t\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial t} \right] \\ g_{tt} \Gamma_{tt}^t &= \frac{1}{2} \left[\frac{\partial g_{tt}}{\partial t} + \frac{\partial g_{tt}}{\partial t} - \frac{\partial g_{tt}}{\partial t} \right] = 0. \end{aligned} \quad (1)$$

So, $\Gamma_{\alpha\beta}^t = 0$ for all α and β .

Christoffel Symbols $\Gamma_{\alpha\beta}^r$

The metric elements are

$$g_{tt} = -1, \quad g_{rr} = 1, \quad g_{\theta\theta} = b^2 + r^2, \quad g_{\phi\phi} = (b^2 + r^2) \sin^2 \theta.$$

Let $\alpha = r$ and $x^\alpha = r$, then

$$g_{r\delta} \Gamma_{\beta\gamma}^\delta = \frac{1}{2} \left[\frac{\partial g_{r\beta}}{\partial x^\gamma} + \frac{\partial g_{r\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial r} \right].$$

Thus, $\delta = r$ and either $\beta = \gamma = \theta$ or $\beta = \gamma = \phi$. So, we have

$$g_{rr} \Gamma_{\theta\theta}^r = \frac{1}{2} \left[\frac{\partial g_{r\theta}}{\partial \theta} + \frac{\partial g_{r\theta}}{\partial \theta} - \frac{\partial g_{\theta\theta}}{\partial r} \right].$$

$$g_{rr} \Gamma_{\phi\phi}^r = \frac{1}{2} \left[\frac{\partial g_{r\phi}}{\partial \phi} + \frac{\partial g_{r\phi}}{\partial \phi} - \frac{\partial g_{\phi\phi}}{\partial r} \right].$$

Therefore, since $g_{rr} = 1$,

$$\Gamma_{\theta\theta}^r = -r, \quad \Gamma_{\phi\phi}^r = -r \sin^2 \theta.$$

Christoffel Symbols $\Gamma_{\alpha\beta}^{\theta}$

The metric elements are

$$g_{tt} = -1, \quad g_{rr} = 1, \quad g_{\theta\theta} = b^2 + r^2, \quad g_{\phi\phi} = (b^2 + r^2) \sin^2 \theta.$$

Let $\alpha = \theta$ and $x^{\alpha} = \theta$, then

$$g_{\theta\delta} \Gamma_{\beta\gamma}^{\delta} = \frac{1}{2} \left[\frac{\partial g_{\theta\beta}}{\partial x^{\gamma}} + \frac{\partial g_{\theta\gamma}}{\partial x^{\beta}} - \frac{\partial g_{\beta\gamma}}{\partial \theta} \right].$$

Thus, $\delta = \theta$. We take $\beta = \theta$ or $\beta = \phi$ due to symmetry. So, we have

$$g_{\theta\theta} \Gamma_{\theta\gamma}^{\theta} = \frac{1}{2} \left[\frac{\partial g_{\theta\theta}}{\partial x^{\gamma}} + \frac{\partial g_{\theta\gamma}}{\partial \theta} - \frac{\partial g_{\theta\gamma}}{\partial \theta} \right].$$

$$g_{\theta\theta} \Gamma_{\phi\gamma}^{\theta} = \frac{1}{2} \left[\frac{\partial g_{\theta\phi}}{\partial x^{\gamma}} + \frac{\partial g_{\theta\gamma}}{\partial \phi} - \frac{\partial g_{\phi\gamma}}{\partial \theta} \right].$$

Nonzero terms occur for $\gamma = r$ in first and $\gamma = \phi$ in second equation.

Christoffel Symbols $\Gamma_{\alpha\beta}^{\theta}$ (cont'd)

Since $g_{\theta\theta} = b^2 + r^2$ and $g_{\phi\phi} = (b^2 + r^2) \sin^2 \theta$, we have

$$g_{\theta\theta}\Gamma_{\theta r}^{\theta} = \frac{1}{2} \left[\frac{\partial g_{\theta\theta}}{\partial r} + \frac{\partial g_{\theta r}}{\partial \theta} - \frac{\partial g_{\theta r}}{\partial \theta} \right].$$

$$(b^2 + r^2)\Gamma_{\theta r}^{\theta} = \frac{1}{2} \frac{\partial g_{\theta\theta}}{\partial r} = r$$

$$\Gamma_{\theta r}^{\theta} = \frac{r}{b^2 + r^2} = \Gamma_{r\theta}^{\theta}.$$

and

$$g_{\theta\theta}\Gamma_{\phi\phi}^{\theta} = \frac{1}{2} \left[\frac{\partial g_{\theta\phi}}{\partial \phi} + \frac{\partial g_{\theta\phi}}{\partial \phi} - \frac{\partial g_{\phi\phi}}{\partial \theta} \right].$$

$$(b^2 + r^2)\Gamma_{\phi\phi}^{\theta} = -\frac{1}{2} \frac{\partial g_{\phi\phi}}{\partial \theta} = -(b^2 + r^2) \sin \theta \cos \theta$$

$$\Gamma_{\phi\phi}^{\theta} = -\sin \theta \cos \theta.$$

Christoffel Symbols $\Gamma_{\alpha\beta}^{\phi}$

The metric elements are

$$g_{tt} = -1, \quad g_{rr} = 1, \quad g_{\theta\theta} = b^2 + r^2, \quad g_{\phi\phi} = (b^2 + r^2) \sin^2 \theta.$$

Let $\alpha = \phi$ and $x^\alpha = \phi$, then

$$g_{\phi\delta} \Gamma_{\beta\gamma}^{\delta} = \frac{1}{2} \left[\frac{\partial g_{\phi\beta}}{\partial x^\gamma} + \frac{\partial g_{\phi\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial \phi} \right].$$

Thus, $\delta = \phi$ and we take $\beta = \phi$ due to symmetry. So, we have

$$\begin{aligned} g_{\phi\phi} \Gamma_{\phi\gamma}^{\phi} &= \frac{1}{2} \left[\frac{\partial g_{\phi\phi}}{\partial x^\gamma} + \frac{\partial g_{\phi\gamma}}{\partial \phi} - \frac{\partial g_{\phi\gamma}}{\partial \phi} \right] \\ &= \frac{1}{2} \frac{\partial g_{\phi\phi}}{\partial x^\gamma}. \end{aligned}$$

Since $g_{\phi\phi} = (b^2 + r^2) \sin^2 \theta$, then $\gamma = r$ or $\gamma = \theta$.

Christoffel Symbols $\Gamma_{\alpha\beta}^{\phi}$ (cont'd)

Since $g_{\phi\phi} = (b^2 + r^2) \sin^2 \theta$, we have

$$g_{\phi\phi} \Gamma_{\phi r}^{\phi} = \frac{1}{2} \frac{\partial g_{\phi\phi}}{\partial r}.$$

$$(b^2 + r^2) \sin^2 \theta \Gamma_{\phi r}^{\phi} = r \sin^2 \theta$$

$$g_{\phi\phi} \Gamma_{\phi\theta}^{\phi} = \frac{1}{2} \frac{\partial g_{\phi\phi}}{\partial \theta}.$$

$$(b^2 + r^2) \sin^2 \theta \Gamma_{\phi\theta}^{\phi} = (b^2 + r^2) \sin \theta \cos \theta$$

Therefore, we have

$$\Gamma_{\phi r}^{\phi} = \frac{r}{b^2 + r^2} = \Gamma_{r\phi}^{\phi}, \quad \Gamma_{\phi\theta}^{\phi} = \cot \theta = \Gamma_{\theta\phi}^{\phi}.$$

Wormhole Metric and Geodesic Equations

$$ds^2 = -dt^2 + dr^2 + (b^2 + r^2)(d\theta^2 + \sin^2 \theta d\phi^2).$$

$$\frac{d^2 t}{d\tau^2} = 0.$$

$$\frac{d^2 r}{d\tau^2} = r \left[\left(\frac{d\theta}{d\tau} \right)^2 + \sin^2 \theta \left(\frac{d\phi}{d\tau} \right)^2 \right].$$

$$\frac{d}{d\tau} \left((b^2 + r^2) \frac{d\theta}{d\tau} \right) = (b^2 + r^2) \sin \theta \cos \theta \left(\frac{d\phi}{d\tau} \right)^2.$$

$$\frac{d}{d\tau} \left((b^2 + r^2) \sin^2 \theta \frac{d\phi}{d\tau} \right) = 0.$$

Christoffel Symbols $\Gamma_{\theta\theta}^r = -r$, $\Gamma_{\phi\phi}^r = -r \sin^2 \theta$, $\Gamma_{\theta r}^\theta = \frac{r}{b^2 + r^2} = \Gamma_{r\theta}^\theta$,
 $\Gamma_{\phi\phi}^\theta = -\sin \theta \cos \theta$, $\Gamma_{\phi r}^\phi = \frac{r}{b^2 + r^2} = \Gamma_{r\phi}^\phi$, $\Gamma_{\phi\theta}^\phi = \cot \theta = \Gamma_{\theta\phi}^\phi$.

Computing $\Gamma_{\beta\gamma}^{\alpha}$ without Lagrangians

Let's compare $\Gamma_{\beta'\gamma'}^{\alpha'}$ in basis $x^{\alpha'}$ to $\Gamma_{\beta\gamma}^{\alpha}$ in basis x^{α} . For $x^{\alpha} = x^{\alpha}(x^{\mu'})$ we define $L_{\mu'}^{\alpha} = \frac{\partial x^{\alpha}}{\partial x^{\mu'}}$. Then, we have (MTW, p. 262),

$$\Gamma_{\beta'\gamma'}^{\alpha'} = L_{\rho}^{\alpha'} L_{\beta'}^{\mu} L_{\gamma'}^{\nu} \Gamma_{\mu\nu}^{\rho} + L_{\mu}^{\alpha'} L_{\beta',\gamma'}^{\mu},$$

where the bases are $e_{\mu'} = L_{\mu'}^{\alpha} e_{\alpha}$ at a given point. So,

$$\Gamma_{\beta'\gamma'}^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^{\rho}} \frac{\partial x^{\mu}}{\partial x^{\beta'}} \frac{\partial x^{\nu}}{\partial x^{\gamma'}} \Gamma_{\mu\nu}^{\rho} + \frac{\partial x^{\alpha'}}{\partial x^{\mu}} \frac{\partial^2 x^{\mu}}{\partial x^{\gamma'} \partial x^{\beta'}}.$$

Since the Christoffel symbols vanish for flat coordinates,

$$\Gamma_{\beta'\gamma'}^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^{\mu}} \frac{\partial^2 x^{\mu}}{\partial x^{\gamma'} \partial x^{\beta'}}.$$

For example, to find $\Gamma_{\beta'\gamma'}^{\alpha'}$ for spherical coordinates, one compute a few derivatives of the spherical coordinate transformations.

Example: Spherical Coordinates

Transformation:

$$x = r \sin \theta \cos \phi,$$

$$y = r \sin \theta \sin \phi,$$

$$z = r \cos \theta.$$

Christoffel Symbols

$$\begin{aligned}\Gamma_{r\theta}^r &= \frac{\partial^2 x}{\partial r \partial \theta} \frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial r \partial \theta} \frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial r \partial \theta} \frac{\partial r}{\partial z} \\ &= \cos \theta \cos \phi \frac{x}{r} + \cos \theta \sin \phi \frac{y}{r} - \sin \theta \frac{z}{r} = 0.\end{aligned}$$

$$\begin{aligned}\Gamma_{\theta\theta}^r &= \frac{\partial^2 x}{\partial \theta^2} \frac{\partial r}{\partial x} + \frac{\partial^2 y}{\partial \theta^2} \frac{\partial r}{\partial y} + \frac{\partial^2 z}{\partial \theta^2} \frac{\partial r}{\partial z} \\ &= -r \sin \theta \cos \phi \frac{x}{r} + r \sin \theta \sin \phi \frac{y}{r} - r \cos \theta \frac{z}{r} = -r.\end{aligned}$$

etc.

Computing Christoffel Symbols in Maple

```
> restart: with( tensor ):
```

Declare coordinates in desired order.

```
> coord := [t, r, theta, phi]:
```

Enter metric components to produce g.

```
> gg:=array( symmetric, sparse, 1..4, 1..4 ):
```

```
gg[1,1] := -1: gg[2,2] := 1: gg[3,3] := r^2+b^2: gg[4,4] := (r^2+b^2)*sin(theta)^2:
```

```
> g := create( [-1,-1], eval(gg) );
```

$$g := \text{table}(\left[\begin{array}{cccc} \text{compts} = & \begin{matrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 + b^2 & 0 \\ 0 & 0 & 0 & (r^2 + b^2) \sin(\theta)^2 \end{matrix} & \right. & \left. \text{index_char} = [-1, -1] \right])$$

Run main routine and display Christoffel symbols (of second kind).

```
> tensorsGR(coord,g,contra_metric,det_met, C1, C2, Rm, Ro, R, G, C):
```

```
> displayGR(Christoffel2,C2);
```

The Christoffel Symbols of the Second Kind

non-zero components :

$$\{2,33\} = -r$$

$$\{2,44\} = -r \sin(\theta)^2$$

$$\{3,23\} = \frac{r}{r^2 + b^2}$$

$$\{3,44\} = -\sin(\theta) \cos(\theta)$$

$$\{4,24\} = \frac{r}{r^2 + b^2}$$

$$\{4,34\} = \frac{\cos(\theta)}{\sin(\theta)}$$