Classical Tests

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We derive the light deflection shift and perhilion shift equations based on the Schwarzschild metric.

Light Deflection

We can use the Schwarschild line element to find the geodesics followed by light rays. The line element is given by

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right).$$

We seek a parametric representation of the light path using the affine parameter, λ. This gives

$$\left(\frac{ds}{d\lambda}\right)^2 = -\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right).$$

We will choose geodesics in the equatorial plane, $\theta = \frac{\pi}{2}$. Then,

$$\left(\frac{ds}{d\lambda}\right)^2 = -\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2.$$

Now we extremize

$$s = \int_{\lambda_1}^{\lambda_2} F(t,\dot{t},\dot{r},\phi,\dot{\phi})\,d\lambda,$$

where

$$F(t,\dot{t},\dot{r},\phi,\dot{\phi}) = -\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2.$$

Since F is independent of t and ϕ variables, we obtain two constants from the geodesics:

$$\left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda} = e \qquad (1)$$

$$r^2\frac{d\phi}{d\lambda} = \ell. \qquad (2)$$

Light rays travel on null geodesics, namely $\frac{ds}{d\lambda} = 0$. Therefore,

$$\begin{aligned} 0 &= -\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2 \\ &= -\left(1-\frac{2M}{r}\right)^{-1}e^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + \frac{\ell^2}{r^2}. \end{aligned}$$

Solving for $\frac{dr}{d\lambda}$, we find

$$\frac{dr}{d\lambda} = \pm\sqrt{e^2 - \frac{\ell^2}{r^2}\left(1 - \frac{2M}{r}\right)}. \tag{3}$$

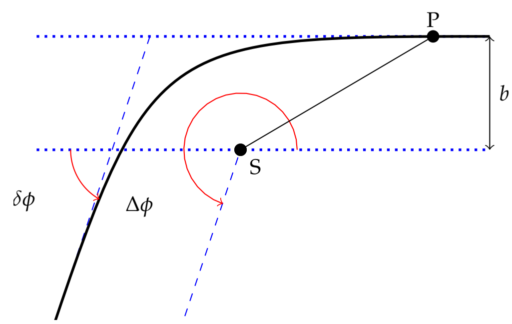

Figure 1: A light ray enters from the right with impact parameter b. It is deflected downward through an angle of $\delta\phi = \Delta\phi - \pi$.

To follow the path of a light ray, we seek a relation between r and ϕ. First, note that

$$\frac{d\phi}{dr} = \frac{d\phi/d\lambda}{dr/d\lambda}.$$

Then,

$$\frac{d\phi}{dr} = \pm\frac{1}{r^2}\left[\frac{e^2}{\ell^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\right]^{-1/2}.$$

Noting that $b = \frac{\ell}{e}$ is the impact parameter, one can integrate this equation to obtain the angle of deflection $\delta\phi = \Delta\phi - \pi = \frac{4M}{b}$. This result follows from assuming that $2M/b$ is small and using the binomial expansion.

Letting $w = b/r$, $dw = -bdr/r^2$,

$$\begin{aligned}
\Delta\phi &= 2\int_{r_1}^{\infty} \frac{dr}{r^2\sqrt{\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)}} \\
&= 2\int_0^{w_1} \frac{dw}{\sqrt{1 - w^2\left(1 - \frac{2M}{b}w\right)}} \\
&= 2\int_0^{w_1} \left(1 - \frac{2M}{b}w\right)^{-1/2} \frac{dw}{\sqrt{\left(1 - \frac{2M}{b}w\right)^{-1} - w^2}} \\
&\approx 2\int_0^{w_1} \left(1 + \frac{M}{b}w\right) \frac{dw}{\sqrt{\left(1 + \frac{2M}{b}w\right) - w^2}},
\end{aligned}$$

where w_1 is a root of

$$0 = 1 - w^2\left(1 - \frac{2M}{b}w\right) = 1 + \frac{2M}{b}w - w^2.$$

We need to evaluate the integral

$$I = \int \frac{1 + aw}{\sqrt{1 + 2aw - w^2}}\, dw$$

where $a = M/b$.

Completing the square, we have

$$1 + 2aw - w^2 = (1 + a^2) - (w - a)^2.$$

This suggests that we make the substitution $w = a + \sqrt{1 + a^2}\sin\theta$. Using the triangle relations in Figure 2, we obtain

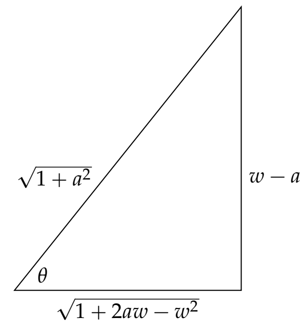

Figure 2: The trigonometric functions of θ can be found for the substitution

$$\sin\theta = \frac{w-a}{\sqrt{1+a^2}}.$$

$$\begin{aligned} I &= \int 1 + a^2 + a\sqrt{1 + a^2}\sin\theta\, d\theta \\ &= (1 + a^2)\theta - a\sqrt{1 + a^2}\cos\theta \\ &= (1 + a^2)\tan^{-1}\left(\frac{w - a}{\sqrt{1 + 2aw - w^2}}\right) - a\sqrt{1 + 2aw - w^2}. \end{aligned}$$

Note that the roots of the radical are $w = a \pm \sqrt{1 + a^2}$ with the positive root $w_1 = a + \sqrt{1 + a^2}$. Then, we can evaluate the needed integral:

$$\begin{aligned} \int_0^{w_1} \frac{1 + aw}{\sqrt{1 + 2aw - w^2}}\, dw &= \left[(1 + a^2)\tan^{-1}\left(\frac{w - a}{\sqrt{1 + 2aw - w^2}}\right) - a\sqrt{1 + 2aw - w^2}\right]_0^{w_1} \\ &= (1 + a^2)\tan^{-1}\left(\frac{w_1 - a}{\sqrt{1 + 2aw_1 - {w_1}^2}}\right) - a\sqrt{1 + 2aw_1 - {w_1}^2} \\ &\quad +(1 + a^2)\tan^{-1}(a) + a \\ &= (1 + a^2)\frac{\pi}{2} + (1 + a^2)\tan^{-1}(a) + a \\ &\approx (1 + a^2)\left(\frac{\pi}{2} + a - \frac{1}{3}a^3\right) + a \\ &\approx \frac{\pi}{2} + 2a. \end{aligned}$$

Then,

$$\Delta\phi = 2(\pi/2 + a.) = \pi + 4M/b. \tag{4}$$

Precession

The geodesic path for Mercury is also determined using

$$\left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = e \tag{5}$$

$$r^2\frac{d\phi}{d\tau} = \ell. \tag{6}$$

However, in this case we need $\left(\frac{ds}{d\tau}\right)^2 = -1$. From the line element, we have

$$-1 = -\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2$$

$$= \quad -\left(1-\frac{2M}{r}\right)^{-1} e^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + \frac{\ell^2}{r^2}.$$

Solving for $\frac{dr}{d\tau}$, we find

$$\frac{dr}{d\tau} = \pm\sqrt{e^2 - \left(1+\frac{\ell^2}{r^2}\right)\left(1-\frac{2M}{r}\right)}$$

$$\frac{d\phi}{dr} = \pm\frac{\ell}{r^2}\left[e^2 - \left(1+\frac{\ell^2}{r^2}\right)\left(1-\frac{2M}{r}\right)\right]^{-1/2}$$

Now we integrate this between the turning points and double that to obtain

$$\begin{aligned}
\Delta\phi &= \pm 2\ell \int_{r_1}^{r_2} \frac{dr}{r^2}\left[e^2 - \left(1+\frac{\ell^2}{r^2}\right)\left(1-\frac{2M}{r}\right)\right]^{-1/2} \\
&= \pm 2\ell \int_{r_1}^{r_2} \frac{dr}{r^2}\left[(e^2-1) + \frac{2M}{r} - \frac{\ell^2}{r^2} + \frac{2M\ell^2}{r^3}\right]^{-1/2} \\
&= \pm 2\ell \int_{r_1}^{r_2} \frac{dr}{r^2}\left[c^2(e^2-1) + \frac{2GM}{r} - \frac{\ell^2}{r^2} + \frac{2GM\ell^2}{c^2 r^3}\right]^{-1/2} \\
&= \mp 2\ell \int_{u_1}^{u_2} du \left[c^2(e^2-1) + 2GMu - \ell^2 u^2 + \frac{2GM\ell^2}{c^2} u^3\right]^{-1/2}.
\end{aligned}$$

Here we have introduced $u = 1/r$ and $du = -dr/r^2$. Having re-inserted factors of G and c, the last term is of order $O(\frac{1}{c^2})$ and is small compared to the other terms.

Neglecting the last term, we can write the argument of the square root in factored form:

$$\Delta\phi \approx 2\int_{u_1}^{u_2} \frac{du}{\sqrt{(u-u_1)(u_2-u)}}.$$

We let $u - u_1 = t^2$, $du = 2t\,dt$. Then, using the trigonometric substitution in Figure 3,

$$\begin{aligned}
\int_{u_1}^{u_2} \frac{du}{\sqrt{(u-u_1)(u_2-u)}} &= 2\int_0^{\sqrt{u_2-u_1}} \frac{dt}{\sqrt{(u_2-u_1)-t^2}} \\
&= 2\int_0^{\pi/2} d\theta = \pi.
\end{aligned}$$

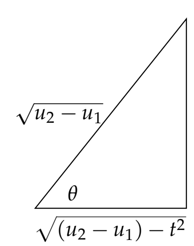

Figure 3: The trigonometric functions of θ can be found for the substitution $t = \sqrt{u_2-u_1}\sin\theta$.

So, we have found that

$$\Delta\phi \approx 2\int_{u_1}^{u_2} \frac{du}{\sqrt{(u-u_1)(u_2-u)}} = 2\pi.$$

Instead, we now consider

$$\Delta\phi \approx 2\int_{u_1^*}^{u_2^*} \frac{du}{\sqrt{(u-u_1)(u_2-u)+\alpha u^3}}.$$

for α small. We expand the integrand around $\alpha = 0$ and keep terms to first order in α.

$$
\begin{aligned}
\frac{1}{\sqrt{(u-u_1)(u_2-u)+\alpha u^3}} &= \frac{1}{\sqrt{(u-u_1)(u_2-u)}} \frac{1}{\sqrt{1+\frac{\alpha u^3}{(u-u_1)(u_2-u)}}} \\
&\approx \frac{1}{\sqrt{(u-u_1)(u_2-u)}} \left[1 - \frac{1}{2}\frac{\alpha u^3}{(u-u_1)(u_2-u)}\right]
\end{aligned}
$$

We have seen that the first term integrates to 2π. So, this leaves

$$
\Delta\phi \approx 2\pi - \alpha \int_{u_1^*}^{u_2^*} \frac{u^3 du}{[(u-u_1)(u_2-u)]^{3/2}}.
$$

Again, we use $u - u_1 = t^2$ and the trigonometric substitution in Figure 3. Then, $u = u_1 + (u_2 - u_1)\sin^2\theta$ and $du = 2(u_2 - u_1)\sin\theta\cos\theta\, d\theta$.

$$
\begin{aligned}
\Delta\phi &\approx 2\pi - \alpha \int_{u_1}^{u_2} \frac{u^3 du}{[(u-u_1)(u_2-u)]^{3/2}} \\
&= 2\pi - \alpha \int_0^{\pi/2} \frac{[u_1 + (u_2-u_1)\sin^2\theta]^3}{[(u_2-u_1)^2\sin^2\theta\cos^2\theta]^{3/2}} 2(u_2-u_1)\sin\theta\cos\theta\, d\theta \\
&= 2\pi - 2\alpha \int_0^{\pi/2} \frac{[u_1\cos^2\theta + u_2\sin^2\theta]^3}{(u_2-u_1)^2\sin^2\theta\cos^2\theta}\, d\theta \\
&= 2\pi - 2\alpha \int_0^{\pi/2} \frac{[u_1\cos^2\theta + u_2\sin^2\theta]^3}{(u_2-u_1)^2\sin^2\theta\cos^2\theta}\, d\theta \\
&= 2\pi - \frac{2\alpha}{(u_2-u_1)^2} \int_0^{\pi/2} \left[u_1^3 \frac{\cos^4\theta}{\sin^2\theta} + 3u_1^2 u_2 \cos^2\theta + 3u_1u_2^2\sin^2\theta + u_2^3\frac{\sin^4\theta}{\cos^2\theta}\right] d\theta \\
&= 2\pi - \frac{2\alpha}{(u_2-u_1)^2}\left[u_1^3\infty + 3(u_1^2u_2 + u_1u_2^2)\frac{\pi}{4} + u_2^3\infty\right] = \infty.
\end{aligned}
$$

So, we need another approximation to avoid such divergent integrals. Returning to the original integral, we have

$$
\begin{aligned}
\Delta\phi &= \pm 2\ell \int_{r_1}^{r_2} \frac{dr}{r^2}\left[c^2(e^2-1) + \frac{2GM}{r} - \frac{\ell^2}{r^2} + \frac{2GM\ell^2}{c^2r^3}\right]^{-1/2}. \\
&= \pm 2\ell \int_{r_1}^{r_2} \frac{dr}{r^2}\left[e^2c^2 - \left(c^2 + \frac{\ell^2}{r^2}\right)\left(1 - \frac{2GM}{rc^2}\right)\right]^{-1/2} \\
&= \pm 2\ell \int_{r_1}^{r_2} \frac{dr}{r^2}\left(1 - \frac{2GM}{rc^2}\right)^{-1/2}\left[e^2c^2\left(1 - \frac{2GM}{rc^2}\right)^{-1} - \left(c^2 + \frac{\ell^2}{r^2}\right)\right]^{-1/2} \\
&= \pm 2\ell \int_{u_1}^{u_2} du \left(1 - \frac{2GM}{c^2}u\right)^{-1/2}\left[e^2c^2\left(1 - \frac{2GM}{c^2}u\right)^{-1} - \left(c^2 + \ell^2u^2\right)\right]^{-1/2} \\
&\approx \pm 2\ell \int_{u_1}^{u_2} du \left(1 + \frac{GM}{c^2}u\right)\left[e^2c^2\left(1 + \frac{2GM}{c^2}u\right) - \left(c^2 + \ell^2u^2\right)\right]^{-1/2}
\end{aligned}
$$

$$
\begin{aligned}
&= \pm 2\ell \int_{u_1}^{u_2} du \left(1 + \frac{GM}{c^2}u\right) \left[(e^2-1)c^2 + 2GMu - \ell^2 u^2\right]^{-1/2} \\
&= 2\int_{u_1}^{u_2} \frac{du}{\sqrt{(u-u_1)(u_2-u)}} + 2\frac{GM}{c^2}\int_{u_1}^{u_2} \frac{u\,du}{\sqrt{(u-u_1)(u_2-u)}} \\
&= 2\pi + \frac{GM}{c^2}\pi(u_1+u_2) \\
&= 2\pi + 2\pi\left(\frac{GM}{c\ell}\right)^2 .
\end{aligned}
$$

Then,

$$
\delta\phi = \Delta\phi - 2\pi = 2\pi\left(\frac{GM}{c\ell}\right)^2 .
$$

Need to find missing term as $\delta\phi = 6\pi\left(\frac{GM}{c\ell}\right)^2 = \frac{6\pi G}{c^2}\frac{M}{a(1-\epsilon^2)}$.