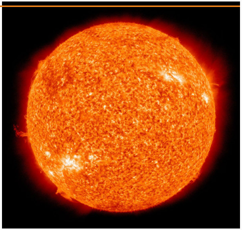
The Lane-Emden-Fowler Equation - An Intriguing ODE

JMM 2023



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Mathematics & Statistics

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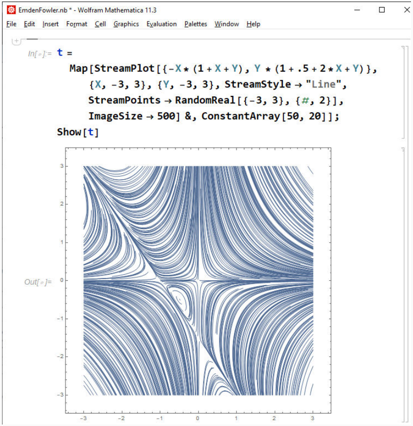


Student Search of a Capstone Problem

- Interested in applications, physics, nonlinear dynamics, solar energy.
- Mancas and Rosu (2015), *Existence of periodic orbits in nonlinear oscillators of Emden-Fowler form*, **Phys. Lett. A 380**, 422-428.
- Paper referred to applications and Generalization of Lane-Emden equation,

$$\begin{aligned} \frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) &= -\theta^n, \\ \theta'' + \frac{2}{\xi}\theta' + \theta^n &= 0. \end{aligned}$$

  - Gravitational potential of self-gravitating gas.
  - Used by Eddington - internal constitution of stars.
  - Thomas-Fermi model of electrons in atoms.
  - Authors studied a related dynamical system.

Literature Search

- **Thomson, W.**, 1862 - Convective Equilibrium - Temperature of Earth. Collected Papers, Vol 5, pg 266.
- **Jonathan Homer Lane** (1819-1880)
  1870 - *On the Theoretical Temperature of the Sun under the Hypothesis of a Gaseous Mass maintaining its Volume by its Internal Heat, and depending on the Laws of Gases as known to Terrestrial Experiment.*

  Independently - A. Ritter (1878), Kelvin (1887).
- **(Jacob) Robert Emden** (1862-1940)
  1907 *Gaskugeln* - thermal behavior of a spherical cloud of gas acting under mutual attraction of molecules.
- **Ralph Howard Fowler** (1889-1944)
  Published papers in 1914, 1930, 1931 on a generalized equation.
- **Subrahmanyan Chandrasekhar** (1910-1995)
  An Introduction to the Study of Stellar Structure, 1939, Univ. Chicago. - Discussed solutions and astrophysical applications.
- **Harold T. Davis**, *Introduction to Nonlinear Differential Equations,* 1962, Dover. - Emphasizes Emden-like equations.

Stellar Structure

- Hydrostatic Equilibrium

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}.$$

- Mass Conservation

$$\frac{dM}{dr} = 4\pi\rho(r)r^2.$$

- Leads to

$$\begin{aligned} \frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP}{dr}\right) &= -G\frac{dM(r)}{dr} \\ &= -4\pi G\rho(r)r^2. \end{aligned}$$

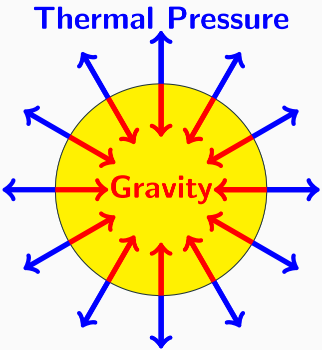


Figure 1: Balance of thermal pressure with gravitational pressure.

Note: Adiabatic $P \propto \rho^\gamma$, $P^{1-\gamma}T^\gamma = \text{const.}$, $T\rho^{1-\gamma} = $const.

Polytropic Models

$$\frac{d}{dr}\left(\frac{r^2}{\rho(r)}\frac{dP}{dr}\right) = -4\pi G\rho(r)r^2.$$

Need an Equation of State, $P = P(\rho)$.

$$P = K\rho^\gamma, \quad \gamma = \frac{n+1}{n},$$

where n is the polytropic index.

- $n = 0$, rocky planets.
- $0.5 \leq n \leq 1$, neutron stars.
- $n = 3$, white dwarfs, Sun.

Gives Poisson Equation: $\nabla^2\rho = f$.

$$\frac{(n+1)K}{4\pi nG}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{\frac{n-1}{n}}}\frac{d\rho}{dr}\right) = -\rho.$$

Lane-Emden Equation

- Rewrite the Poisson Equation:

$$\frac{(n+1)K}{4\pi nG}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{\frac{n-1}{n}}}\frac{d\rho}{dr}\right) = -\rho.$$

- Let $\rho = \rho_c\theta^n$, $r = \alpha\xi$, then get Lane-Emden Equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n.$$

- Initial Conditions, $\theta(0) = 1, \theta'(0) = 0.$
- Fowler generalized to Emden-Fowler Equation

$$\frac{d}{d\xi}\left(\xi^\rho\frac{d\theta}{d\xi}\right) + b\xi^\sigma\theta^n = 0.$$

$$\alpha = \left(\frac{(n+1)K}{4\pi G}\rho_c^{\frac{1}{n}-1}\right)^{\frac{1}{2}}$$

$n=3$	Eddington
$n=\frac{3}{2}$	Thomas-Fermi Model
$n=\frac{3}{2},\frac{5}{2}$	Kelvin, "Homer Lane's Function,"

Generalizations: $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + f(y) = 0.$

- $f(y) = (y^2 - C)^{\frac{3}{2}}$, Chandrasekhar - white dwarfs
- $f(y) = e^y$, - isothermal gas spheres, Emden–Chandrasekhar equation.

Known Solutions: $\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n,\ \theta(0) = 1,\ \theta'(0) = 0.$

- $n = 0,\ \theta'' + \dfrac{2}{\xi}\theta' + 1 = 0.$ **Solution:** $\theta = -\dfrac{1}{6}\xi^2 - \dfrac{c_1}{\xi} + c_2.$
- $n = 1,\ \theta'' + \dfrac{2}{\xi}\theta' + \theta = 0.$ **Solution:** $\theta = \dfrac{c_1 \cos\xi + c_2 \sin\xi}{\xi}.$
- Apply conditions: $n = 0 : \theta = 1 - \dfrac{\xi^2}{6},\ n = 1 : \theta = \dfrac{\sin\xi}{\xi}.$
- $n = 5,\ \theta = \dfrac{1}{\sqrt{1 + \frac{1}{3}\xi^2}}.$
- Series: $\theta = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 - \cdots,\ \xi < 1.$
- Sambhunath Srivastava (1962), $n = 5,\ \theta = \dfrac{\sin\left(\ln\sqrt{\xi}\right)}{\sqrt{\xi}\left[3 - 2\sin^2\left(\ln\sqrt{\xi}\right)\right]}.$

Numerical Solutions

Write $\theta'' = -\frac{2}{\xi}\frac{d\theta}{d\xi} - \theta^n$.

For $\theta_i \approx \theta(\xi_i)$, and $\omega_i = \left(\frac{d\theta}{d\xi}\right)_i$, $\xi_i = i\Delta\xi, i = 0, 1, \ldots, N$,

$$\begin{aligned} \theta_{i+1} &= \theta_i + \Delta\xi\omega_i \\ \omega_{i+1} &= \omega_i + \Delta\xi\left[\frac{2}{\xi}_i\omega_i - \theta_i^n\right]. \end{aligned}$$

Set conditions $\theta_0 = 1$, $\omega_0 = 0$.

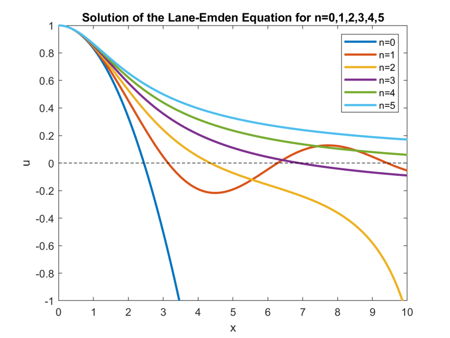


Figure 2: Numerical Solutions for $n = 0, 1, 2, 3, 4, 5$. **Note** - Zeros are important for $n < 5$.

Nonlinear Dynamics

Mancas and Rosu [9] studied the Emden-Fowler equation ($\theta \to \eta$),

$$\frac{d}{d\xi}\left(\xi^2 \frac{d\eta}{d\xi}\right) = \alpha \xi^\lambda \eta^n,$$

Using a homework problem in Jordan and Smith [7], one sets[1]

$$X = \frac{\xi \eta'}{\eta} \quad \text{and} \quad Y = \xi^{\lambda-1} \frac{\eta^n}{\eta'}$$

and $\xi = e^t$, to obtain a two-dimensional autonomous system,

$$\begin{aligned} \dot{X} &= -X(1 + X - \alpha Y) \\ \dot{Y} &= Y(1 + \lambda + nX - \alpha Y). \end{aligned} \tag{1}$$

Here the dot represents $\frac{d}{dt}$.

[1] Chandrasekhar studied homology invariant functions for Lane-Emden eqn., pg 105, $u = -\frac{\xi \eta^{n\,\prime}}{\eta}$, $v = \frac{-\xi \eta'}{\eta}$. Then, $\frac{u}{v}\frac{dv}{du} = \frac{u + v - 1}{u + nv - 3}$, attributed to E. A. Milne.

Linear Stability Analysis ($n \neq 1$)

$$\begin{aligned}\dot{X} &= -X(1+X-\alpha Y)\\ \dot{Y} &= Y(1+\lambda+nX-\alpha Y).\end{aligned} \tag{2}$$

The Jacobian matrix,

$$J(X,Y) = \begin{pmatrix} -1-2X+\alpha Y & \alpha X \\ nY & 1+\lambda+nX-2\alpha Y \end{pmatrix},$$

with eigenvalues $\mu = \dfrac{\text{tr}J \pm \sqrt{\text{tr}^2 J - 4\det J}}{2} \equiv \dfrac{\text{tr}J \pm \sqrt{\Delta}}{2}$.

(X_0, Y_0)	det J	tr J	Δ
$(0,0)$	$-1-\lambda$	λ	$(\lambda+2)^2$
$(-1,0)$	$\lambda-n+1$	$\lambda-n+2$	$(\lambda-n)^2$
$(0, \dfrac{1+\lambda}{\alpha})$	$-\lambda(1+\lambda)$	-1	$(1+2\lambda)^2$
$\left(\dfrac{\lambda}{1-n}, \dfrac{\lambda-n+1}{\alpha(1-n)}\right)$	$\dfrac{\lambda(n-\lambda-1)}{n-1}$	$\dfrac{2\lambda-n+1}{n-1}$	$\dfrac{4\lambda n(\lambda-n+1)+(-1+n)^2}{(n-1)^2}$

The Behavior of the Equilibrium Points - Standard Diagram

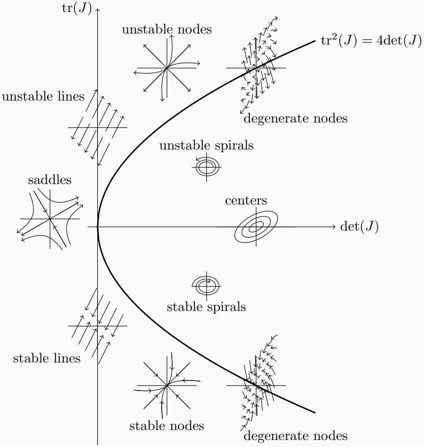
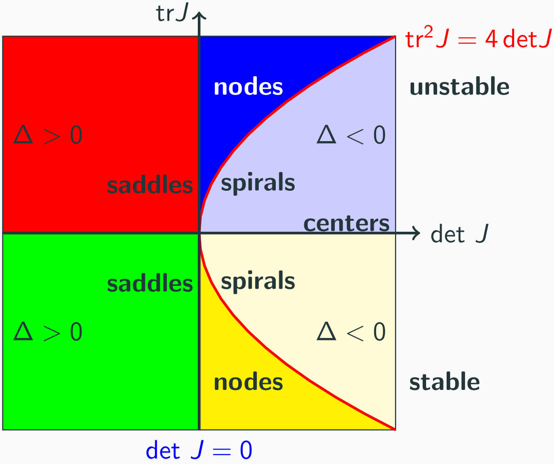


Figure 3: The behavior of equilibrium points in the det − tr plane. The curve $\mathrm{tr}^2(J) = 4\det(J)$ indicates where the discriminant vanishes.

Stability Analysis - Color Coded

The stability of the equilibrium points depends on the eigenvalues, which in turn depend on the trace and determinant of J.



Stability for $X = 0, Y = 0$

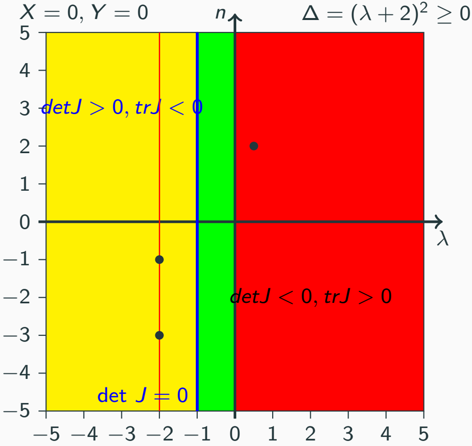


Figure 4: Blue line: det $J = 0$. Red line: $\Delta = 0$. The equilibrium point is a saddle or a stable node. The black dots are examples in [9] for $\alpha = -1$, and $(n, \lambda) = (-2, -3)$, $(n, \lambda) = (-2, -1)$, and $(n, \lambda) = (\frac{1}{2}, 2)$.

Stability for $X = -1, Y = 0$

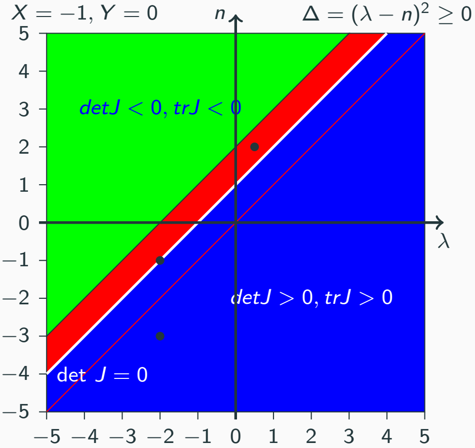


Figure 5: Stability for the equilibrium point $(X, Y) = (-1, 0)$. The white line is where $\det J = 0$. The red line is for $\Delta = 0$. The diagram indicates the equilibrium point can only be a saddle or an unstable node.

Stability for $(X,Y)=\left(\frac{\lambda}{1-n},\frac{\lambda-n+1}{\alpha(1-n)}\right)$.

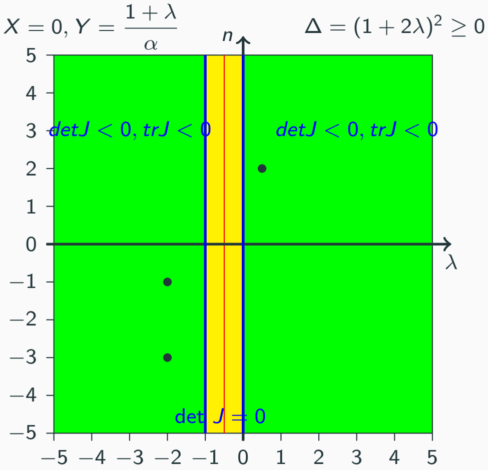


Figure 6: The blue lines show where $\det J=0$. The red line is where $\Delta=0$.

The equilibrium point can only be a saddle or a stable node.

Stability for $(X, Y) = \left(\frac{\lambda}{1-n}, \frac{\lambda-n+1}{\alpha(1-n)}\right)$.

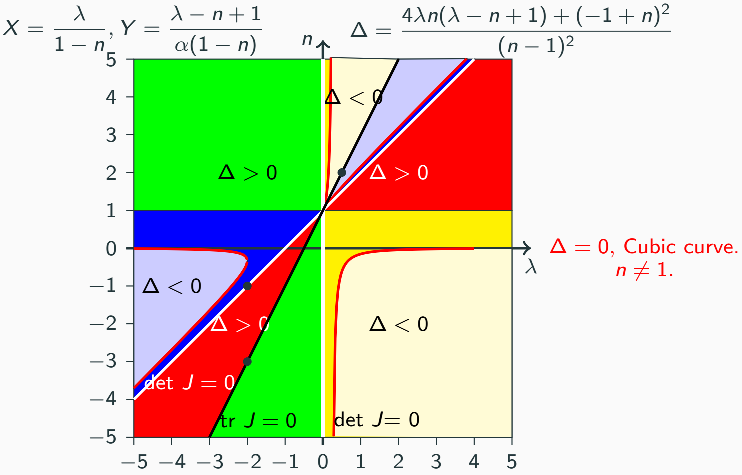
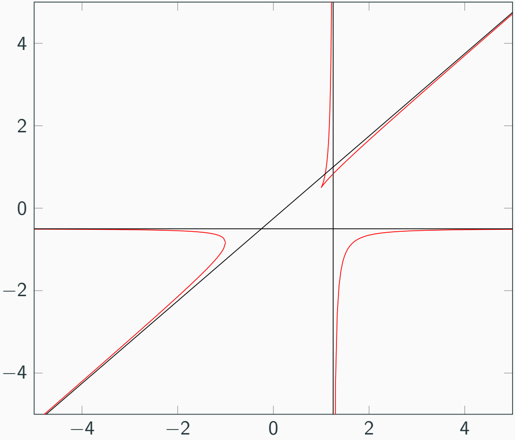


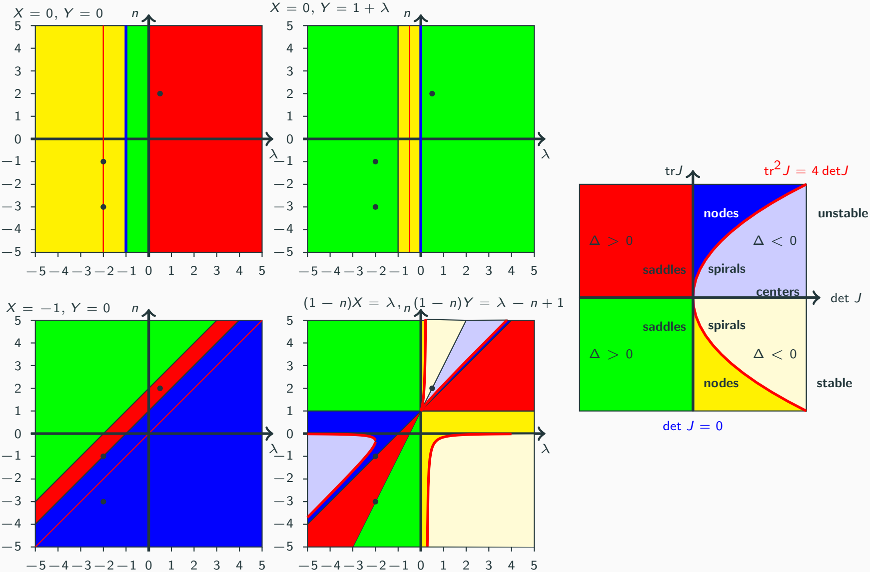
Figure 7: White lines: det $J = 0$. Red curves: $\Delta = 0$. Equilibrium point is a stable spiral (light yellow), unstable spiral (light blue), or center (black line).

Cubic Curves $x = \lambda + 1, y = n - \frac{1}{2}$

$$p(x, y) = -4\left(y - x - \frac{1}{4}\right)\left(y + \frac{1}{2}\right)\left(x - \frac{5}{4}\right) - \frac{3}{4}y + \frac{5}{8}.$$

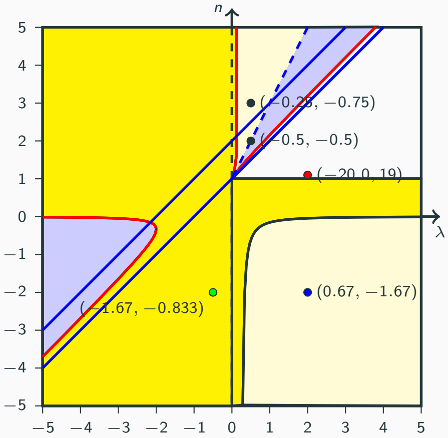


Stability of Equilibria

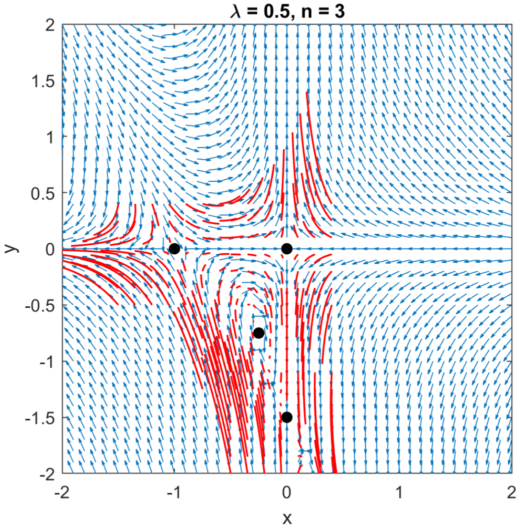


Summary of Analysis

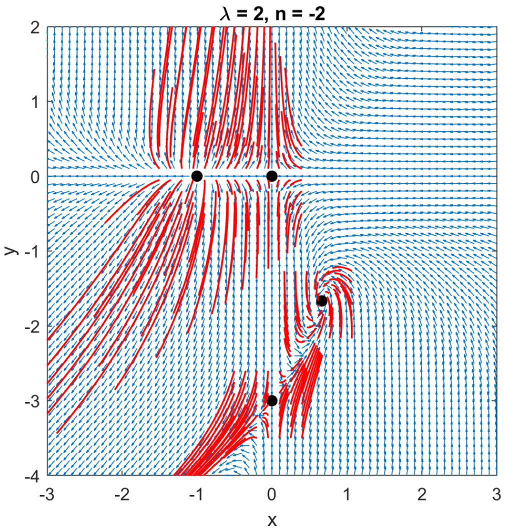
- How does local behavior connect to global behavior?
- Can only be done numerically. Need parameters for interesting behaviors.
- Mancas and Rosu [9] concerned periodic solutions (centers).



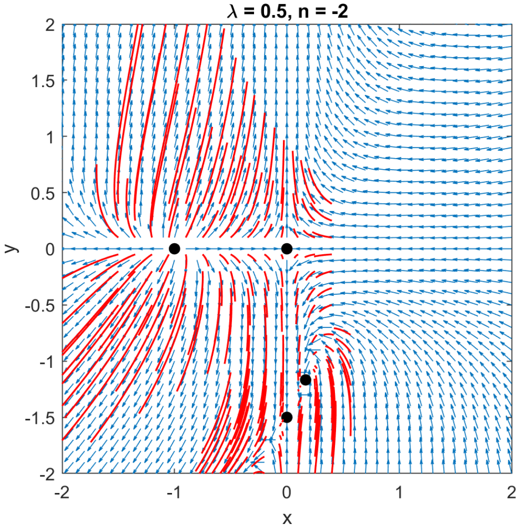
Phase Portrait for $n = 3$, $\lambda = 0.5$ - Stable Spiral



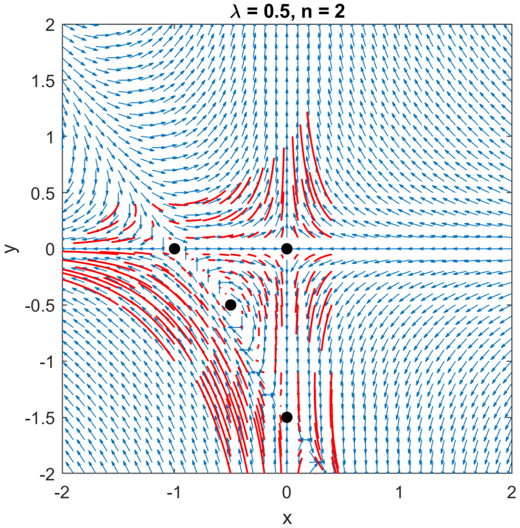
Phase Portrait for $n = -2$, $\lambda = 2$ - Stable Node



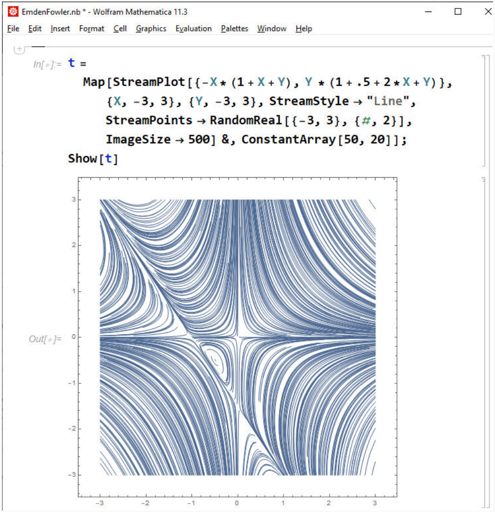
Phase Portrait for $n = -2$, $\lambda = 0.5$ - Stable Node



Phase Portrait for $n = 2$, $\lambda = 0.5$ - Center



Mathematica - Detailed Plot $n = 2$, $\lambda = 0.5$ - Center



Summary

Convert Emden-Fowler equation $\frac{d}{d\xi}\left(\xi^2 \frac{d\eta}{d\xi}\right) = \alpha \xi^\lambda \eta^n$, using homology invariant functions,

$$X = \frac{\xi \eta'}{\eta} \quad \text{and} \quad Y = \xi^{\lambda-1} \frac{\eta^n}{\eta'}$$

and $\xi = e^t$, to obtain a two-dimensional autonomous system,

$$\begin{aligned} \dot{X} &= -X(1 + X - \alpha Y) \\ \dot{Y} &= Y(1 + \lambda + nX - \alpha Y). \end{aligned}$$

Using color-coding, we can identify interesting dynamics involving nodes, saddles, and spirals or centers.

Thanks for listening!

References i

- Davis, H. T. *Introduction to Nonlinear Differential Equations,* Dover, (1962).
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