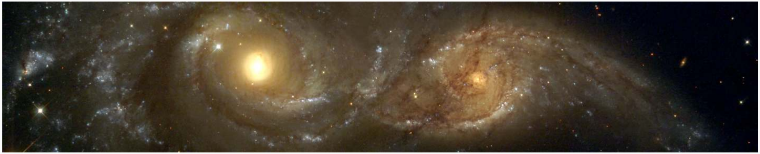

Introduction to Relativity
PHY 490, Fall 2023

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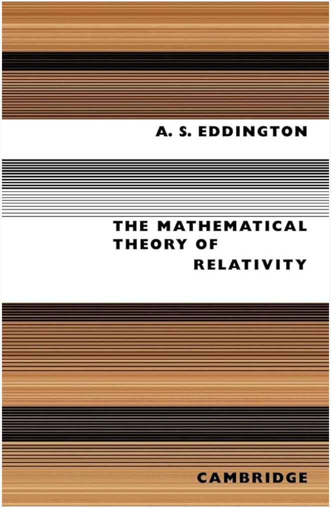
Table of Contents



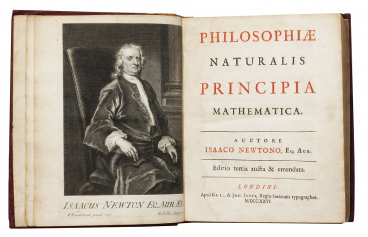
Isaac Newton (1642-1727)

In 1680s Newton sought derivation of Kepler's planetary laws of motion.

- Principia 1687.
- Took 18 months.
- Laws of Motion.
- Law of Gravitation.
- 1759 - Halley's Comet

Objects on the Earth feel same force as the planets orbiting the sun.

$$F = G\frac{mM}{r^2}.$$



John Michell (1724-1793) - restored from obscurity

- Natural philosopher, clergyman
- Applied Newton's Corpuscular Theory.
- *Philosophical Transactions of the Royal Society of London*, 1783.
- A star's gravitational pull might be so strong that the escape velocity would exceed the speed of light!
  - **Dark Stars**.
- Pierre-Simon Laplace (1749-1827), *Exposition du Système du Monde* - 1796
- Consider **escape velocity**.

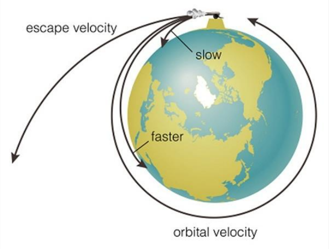


Figure 1: Firing projectiles.

Escape Velocity from $E = T + U$

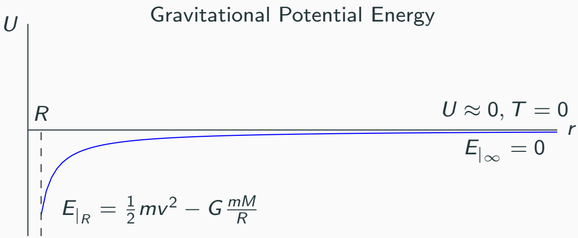
- Kinetic energy: $T = \frac{1}{2}mv^2$.
- Potential energy:

$$U = \int_\infty^R F(\rho)\, d\rho = \int_\infty^R G\frac{mM}{\rho^2}\, d\rho = -G\frac{mM}{R}.$$

- Escape velocity:
  Energy conservation.

$$\frac{1}{2}mv^2 - G\frac{mM}{R} = 0.$$

$$v = \sqrt{\frac{2GM}{R}}.$$



Common Escape Velocities, $v = \sqrt{\frac{2GM}{R}}$,

Escape rates for some celestial bodies, $G = 6.67 \times 10^{-11} Nm^2/kg^2$.

	Mass M (kg)	Radius R (m)	Escape Velocity v (m/s)
Moon	7.348×10^{22}	1.737×10^{6}	2,376 (5,300 mph)
Earth	5.972×10^{24}	6.378×10^{6}	11,176 (25,000 mph)
Jupiter	1.898×10^{27}	7.1492×10^{7}	59,511 (133,000 mph)
Sun	1.989×10^{30}	6.957×10^{8}	617,567 (1.38 million mph)

For light, $R = \frac{2GM}{c^2}$,
$v = c = 3.0 \times 10^8$ m/s.

- Earth, $R = .0088$ m.
- Sun, $R = 2.9$ km,
- $\frac{\text{Sun Mass}}{\text{Earth Mass}} = 3.3 \times 10^5$

But, light is a wave!

Density: $\rho \sim M/R^3$.
Light fails to escape when

$$M \sim (c^2/G)^{3/2} \rho^{-1/2}$$

For lead, $\rho \sim 5000$ kg-m^{-3},
$M \sim 7.01 \times 10^{38} kg = 3.5 \times 10^8 M_\odot$.
Then, smallest mass would be

$$M \sim 10^8 (\rho_*/\rho)^{1/2} M_\odot.$$

James Clerk Maxwell (1831-1879) - Light = EM Wave

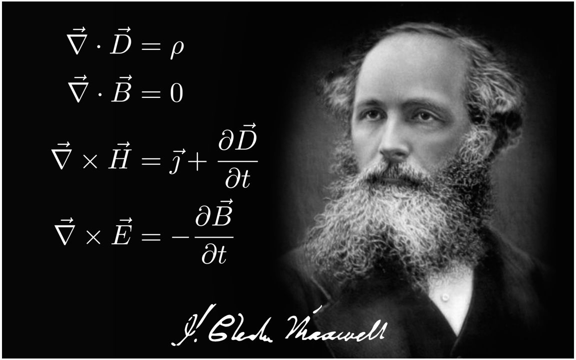


Figure 2: Equations of Electricity and Magnetism
Gauss' Law, No magnetic monopoles, Maxwell-Ampere Law, Faraday's Law.

... and then came A. Einstein!

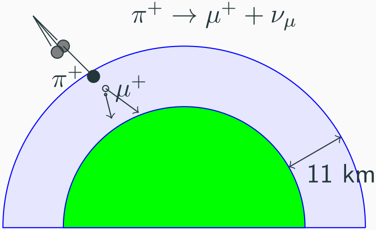
1905 - Einstein's Miracle Year

- Photoelectric effect (March/June).
- Brownian motion (May/July).
- Special Relativity (June/September).
  - Inspired by Maxwell's Theory.
  - Two Postulates
    - Physics is same for all inertial observers.
    - Speed of light same for everyone.
  - Consequences.
    - Time dilation.
    - Length contraction.
    - Space and Time relative.
- $E = mc^2.(September/November)$

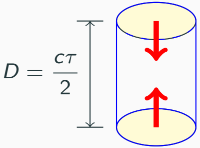
Figure 3: Einstein (1879-1955)

Time Dilation - Moving clocks tick slower.

- Examples -
  - Plane trip
    - 620 mph (277 m/s)
    - Lose 3 ns/hr.
  - Muon
  - Cosmic rays collide with nuclei.
  - Pions decay into muons.
    - Lifetime 2.2 μs
    - At 0.995c, travels 660 m

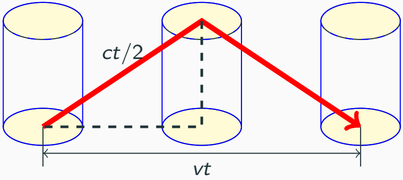


Light clock



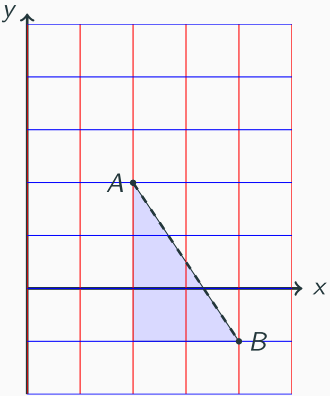
Moving light clock

$$t = \frac{\tau}{\sqrt{1 - \frac{v^2}{c^2}}}$$

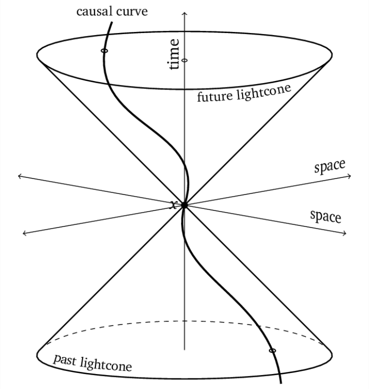


Space, Time, and Spacetime

From René Descartes:

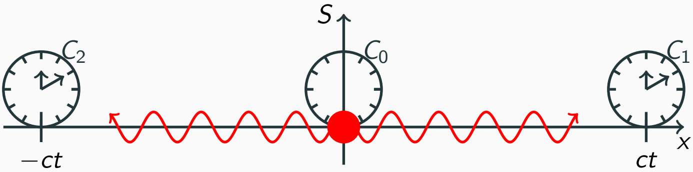


From Hermann Minkowski:

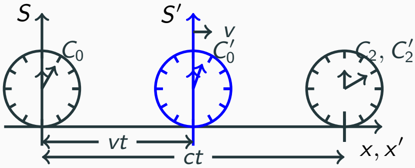


Particles move in straight lines to maximize lifetime.

Lorentz Transformation



- Clock C_0, synchronized with C_1, C_2.
- Pulse sent at at $t = 0$.
- Travels ct to C_1, C_2.
- Then, $x = \pm ct$, or $x^2 - c^2t^2 = 0$.
- System S' travels v w.r.t. S.
- $x'^2 - c^2t'^2 = 0$.
- $\Delta x^2 - c^2\Delta t^2 = \Delta x'^2 - c^2\Delta t'^2$.

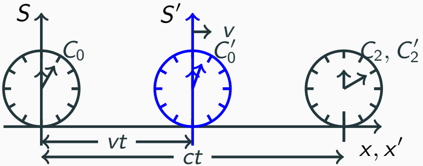


Lorentz Transformation (con't)

- $\Delta x^2 - c^2\Delta t^2 = \Delta x'^2 - c^2\Delta t'^2$.
- $\Delta x' = 0$, C_0' at rest w.r.t. S'.
- According to S, C_0' at $x = vt$.

$$\begin{aligned} x^2 - c^2t^2 &= -c^2t'^2 \\ (v^2-c^2)t^2 &= -c^2t'^2 \\ t &= \gamma t' \\ \gamma &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}. \end{aligned}$$

- Galilean transformation, $x = x' + vt', t = t'$.
- Assume Lorentz transformation, $x = ax' + bct', t = \gamma t'$.



- $x' = 0, x = vt \Rightarrow vt = bc\gamma^{-1}t$. So, $b = \beta\gamma$, $\beta = v/c$.
- $x = 0, x' = -vt \Rightarrow$ $0 = -avt' + bct'$, or $a = \gamma$.
- Thus, $x = \gamma\left(x' + \beta ct'\right)$.
- $t = x/c, t' = x'/c \Rightarrow$ $ct = \gamma(ct' + \beta x')$.

Lorentz Transformation Summary

The Lorentz transformation in 1+1 dimensional spacetime is

$$\begin{aligned} x &= \gamma(x' + vt') = \gamma(x' + \beta ct'), && (1) \\ ct &= c\gamma(t' + \frac{vx'}{c^2}) = \gamma(ct' + \beta x'), && (2) \end{aligned}$$

with Lorentz factor $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}}$, $\beta = \frac{v}{c}$.

The inverse transformation is

$$\begin{aligned} x' &= \gamma(x - vt) = \gamma(x - \beta ct), && (3) \\ ct' &= c\gamma(t - \frac{vx}{c^2}) = \gamma(ct - \beta x). && (4) \end{aligned}$$

This is also referred to as a Lorentz boost.

Matrix Representation

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}$$
$$= \begin{pmatrix} \cosh\chi & \sinh\chi \\ \sinh\chi & \cosh\chi \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix}$$

Here $\beta = \tanh\chi$, $\gamma = (1-\beta^2)^{-1/2} = \cosh\chi$, where χ is called the rapidity. The inverse transformation is given by

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \cosh\chi & \sinh\chi \\ \sinh\chi & \cosh\chi \end{pmatrix}^{-1} \begin{pmatrix} x \\ ct \end{pmatrix}$$
$$= \begin{pmatrix} \cosh\chi & -\sinh\chi \\ -\sinh\chi & \cosh\chi \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}$$

Group Structure of Lorentz Boost

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \cosh\chi & -\sinh\chi \\ -\sinh\chi & \cosh\chi \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix} \equiv \Lambda(\chi) \begin{pmatrix} x \\ ct \end{pmatrix}.$$

- Composition $\Lambda(\chi_1)\Lambda(\chi_2) = \Lambda(\chi_1 + \chi_2)$.

$$\begin{pmatrix} \cosh\chi_1 & -\sinh\chi_1 \\ -\sinh\chi_1 & \cosh\chi_1 \end{pmatrix} \begin{pmatrix} \cosh\chi_2 & -\sinh\chi_2 \\ -\sinh\chi_2 & \cosh\chi_2 \end{pmatrix} = \begin{pmatrix} \cosh(\chi_1+\chi_2) & -\sinh(\chi_1+\chi_2) \\ -\sinh(\chi_1+\chi_2) & \cosh(\chi_1+\chi_2) \end{pmatrix}$$

- Addition of Velocities:

$$\begin{aligned} \tanh\chi &= \frac{\tanh\chi_1 + \tanh\chi_2}{1 + \tanh\chi_1 \tanh\chi_2} \\ v &= \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}. \end{aligned}$$

- Identity ($\chi = 0.$), Inverse, Associative.
- Similar to (imaginary) rotation group.

Addition of Velocities

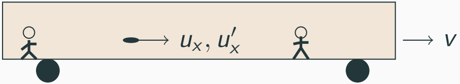
A passenger fires a bullet at $0.6c$ relative to a train moving at $0.8c$. How fast is the bullet moving relative to the ground? It is not $1.4c$.

Another derivation:

$$\begin{aligned} dx &= \gamma(dx' + \beta c dt') &= \gamma(u_x' + v)dt', \\ dt &= \gamma\left(dt' + \frac{v}{c^2}dx'\right) &= \gamma\left(1 + \frac{vu_x'}{c^2}\right)dt'. \end{aligned}$$

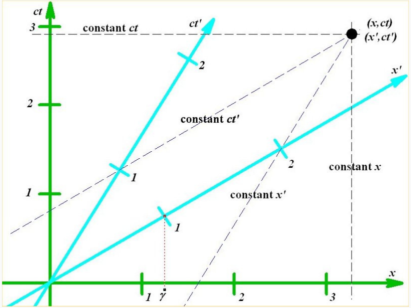
So,

$$u_x = \frac{dx}{dt} = \frac{u_x' + v}{1 + \frac{vu_x'}{c^2}}$$



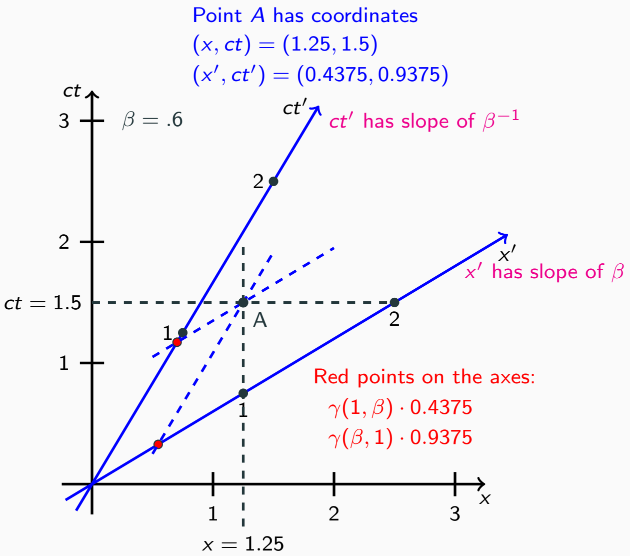
Minkowski Diagrams

- Reference frame $S : (x, ct)$.
- Reference frame $S' : (x', ct')$.
- x'-axis: $x' = 1$, $ct' = 0$.
- Then, $x = \gamma$, $ct = \beta\gamma$.
- Thus, $ct = \beta x$.
- x'-axis has slope $\beta = v/c$
- ct' axis: $x' = 0$, $ct' = 1$.
- Then, $ct = \gamma$, $x = \gamma\beta = \beta ct$.
- Thus, ct'-axis has slope $1/\beta = c/v$.

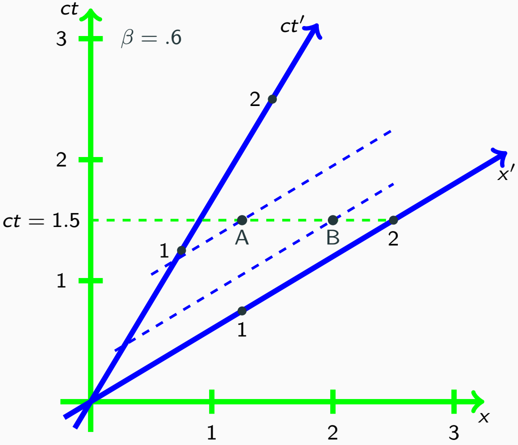


In Figure $\beta = 0.6$. Thus, $\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{5}{4}$.
From $x = \gamma$ and $ct = \beta\gamma$, locate the $(1, 0)$ in the primed system.

Reading Coordinates on a Minkowski Diagram

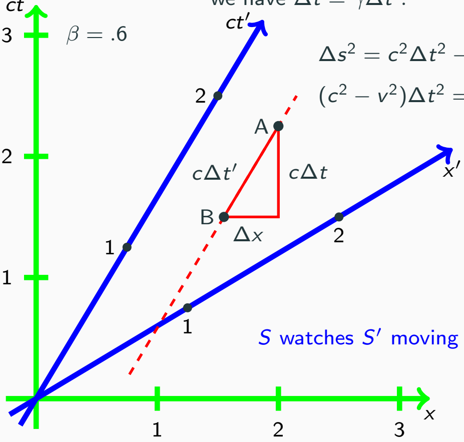


Simultaneity



Time Dilation

Using $c\Delta t = \gamma(c\Delta t' + \beta \Delta x')$ and $\Delta x' = 0$,
we have $\Delta t = \gamma \Delta t'$.



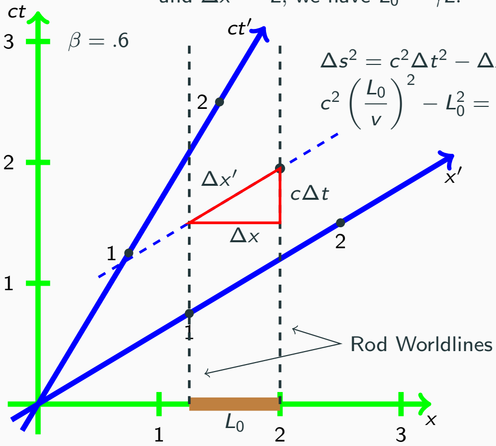
$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 = c^2 \Delta t'^2$$

$$(c^2 - v^2)\Delta t^2 = c^2 \Delta t'^2 \Rightarrow \Delta t = \gamma \Delta t'.$$

S watches S' moving from A to B.

S' looks at a rod at rest in the S-frame.

Using $\Delta x = \gamma(\Delta x' + \beta c\Delta t')$, $c\Delta t' = 0$, $\Delta x = L_0$ and $\Delta x' = L$, we have $L_0 = \gamma L$.

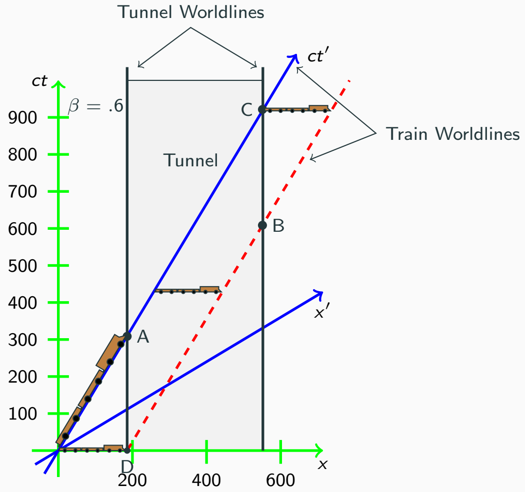


$$\Delta s^2 = c^2\Delta t^2 - \Delta x^2 = -\Delta x'^2$$

$$c^2\left(\frac{L_0}{v}\right)^2 - L_0^2 = -L^2 \Rightarrow L_0 = \gamma L.$$

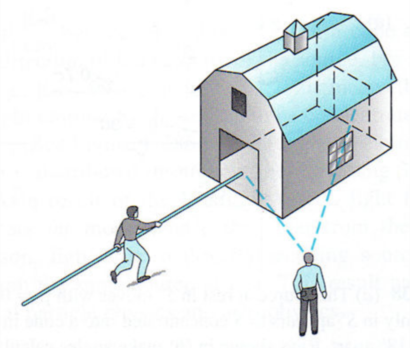
Train in Tunnel Problem

A relativistic train of rest length 240 meters travels at 0.6c through a tunnel which has rest length 360 meters.



Pole in the Barn

A farmer sees a man and the 10m pole contract to 5m. Therefore, the whole pole fits inside the 5m barn. But, the running man sees the 5m barn contract to 2.5m. Therefore, the front of the pole exits the barn before its end is inside the barn. Who is right?



Twin Paradox

Homer and Ulysses are identical twins. Ulysses travels at $0.8c$ to a distant star and returns to Earth while Homer remains at home.

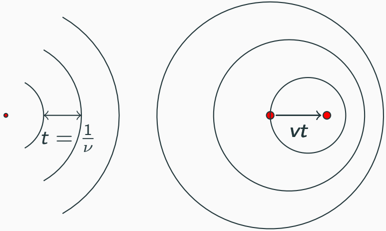
The round trip took Ulysses 6 years. When he returned, he found that Homer aged 10 years.

However, if motion is relative, Ulysses thinks he was at rest and Homer went away. In this case, Homer aged 3.6 years while Ulysses aged 6 years.

Who is right?

Doppler Effect for a Moving Source

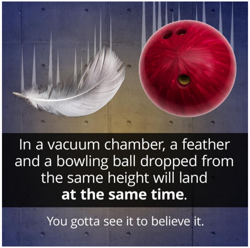
- Classical Doppler: $\lambda' = \frac{c}{\nu} - vt = \frac{c}{\nu}(1-\beta)$.
  Apparent frequency: $\nu' = \frac{c}{\lambda'} = \frac{\nu}{1-\beta}$.

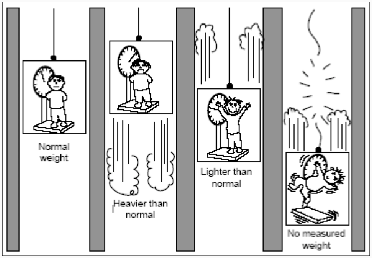


- Relativistic Doppler: Source clock ticks slower, $\nu \to \nu/\gamma$.
  Apparent frequency: $\nu' = \frac{\nu}{\gamma(1-\beta)} = \nu\sqrt{\frac{1+\beta}{1-\beta}}$.
- Galaxy moves away ($\beta < 0$) - **redshift** ($\nu' < \nu$ and $\lambda' > \lambda$).

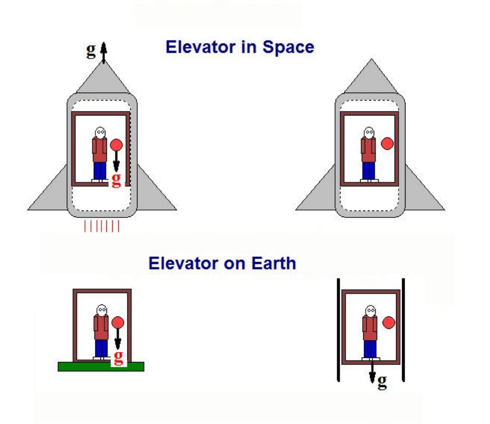
Einstein's Happiest Thought

- Einstein spent years generalizing Special Relativity.
- Galileo - Everything falls at the same rate.
- Einstein - When you fall freely, gravity disappears.
- Led to the Equivalence Principle.





The Equivalence Principle



There are no (local) experiments which can distinguish non-rotating free fall under gravity from uniform motion in space in the absence of gravity.

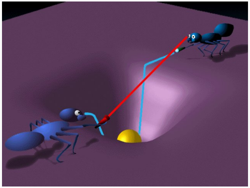
General Relativity - 1915

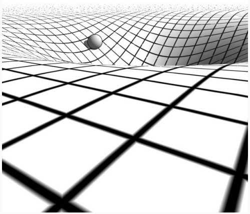
Einstein generalized special relativity to Curved Spacetime.

- Einstein's Equation.
- Gravity = Geometry

$$G_{\mu\nu} = 8\pi T_{\mu\nu}.$$

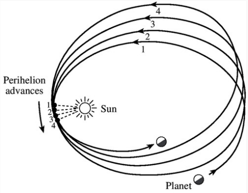
- Mass tells space how to bend and space tell mass how to move.
- Predictions. (Wheeler)
  - Perihelion Shift of Mercury.
  - Bending of Light.
  - Time dilation.





Classical Tests - Perihelion Shift of Mercury

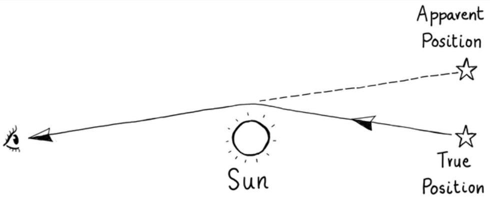
- First noted by Le Verrier, 1859. 38″ (arc seconds) per century.
- Re-estimated by Newcomb, 1882.
- Ellipse axis shifts 43″ per century.



arcsec/cent	**Cause**
532.3035	Gravitational tugs by other bodies
0.0286	Oblateness of Sun
42.9799	General Relativity
-0.0020	Lense-Thirring
575.31	Total Predicted
574.10 ± 0.65	Observed

Classical Tests - Deflection of Light

- Deflection of light - when light passes near a large mass its path is slightly bent.
- 1919 Eclipse observed an island near Brazil and near the west coast of Africa.

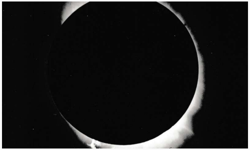



LIGHTS ALL ASKEW IN THE HEAVENS

Men of Science More or Less Agog Over Results of Eclipse Observations.

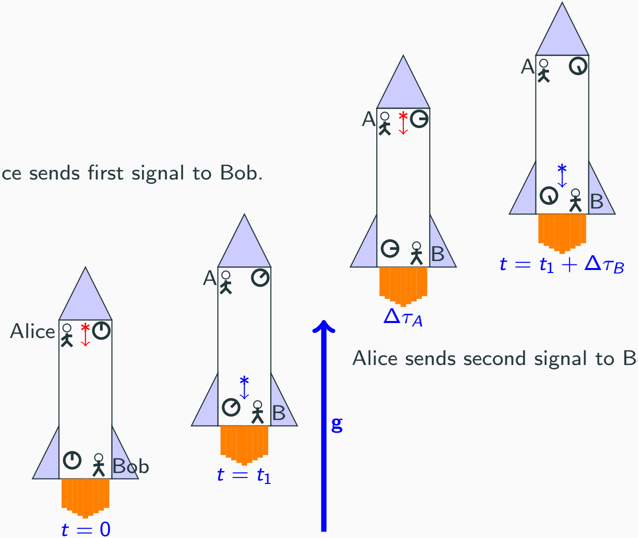
EINSTEIN THEORY TRIUMPHS

Stars Not Where They Seemed or Were Calculated to be, but Nobody Need Worry.



Classical Tests - Gravitational Time Dilation

Alice sends first signal to Bob.



Alice sends second signal to Bob.

Derivation of Gravitational Time Dilation

- Bob and Alice's positions for accelerating rocket:
  $z_B(t) = \frac{1}{2}gt^2, \qquad z_A(t) = h + \frac{1}{2}gt^2.$
- Pulse emitted at $t = 0$ and received at $t_1 : z_A(0) - z_B(t_1) = ct_1.$
- Second pulse emitted travels distance
  $z_A(\Delta\tau_A) - z_B(t_1 + \Delta\tau_B) = c(t_1 + \Delta\tau_B - \Delta\tau_A)$
- Assume $\Delta\tau_A$ small, we have

$$\begin{aligned} h - \frac{1}{2}gt_1^2 &= ct_1, \\ h - \frac{1}{2}gt_1^2 - gt_1\Delta\tau_B &= c(t_1 + \Delta\tau_B - \Delta\tau_A). \end{aligned} \tag{5}$$

- Assume gh/c^2 small, $t_1 \approx h/c$ and

$$\Delta\tau_B = \Delta\tau_A \left(1 - \frac{gh}{c^2}\right).$$

Gravitational Redshift

The time interval for received pulses is smaller

$$\Delta\tau_B = \Delta\tau_A \left(1 - \frac{gh}{c^2}\right).$$

In general, note $gh = \Phi_A - \Phi_B$ is gravitational potential difference. Then, the rate of emission and reception, $1/\Delta\tau$, is

$$\omega_B = \left(1 - \frac{\Phi_A - \Phi_B}{c^2}\right)^{-1} \omega_A \approx \left(1 + \frac{\Phi_A - \Phi_B}{c^2}\right) \omega_A$$

For a star of radius R and signal received far away, and noting $\Phi_A - \Phi_B = \frac{GM}{r_B} - \frac{GM}{r_A}$, we have the **gravitational redshift**

$$\omega_\infty = \left(1 - \frac{GM}{Rc^2}\right) \omega_{\text{star}}.$$

Time Dilation and GPS

Gravitational redshift - clocks in a gravitational field observed from a distance tick slower. (1960s, Pound-Rebka-Snider experiments)

- Special Relativity.

$$\delta t = \frac{\delta \tau}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

- General Relativity.

$$\begin{aligned} \delta t &= \delta \tau \sqrt{1 - \frac{2GM}{rc^2}} \\ &\approx \delta \tau \left(1 - \frac{GM}{rc^2}\right). \end{aligned}$$

- Application - GPS

GPS Satellites

- Global Positioning System
- 32 Satellites (max)
- Semi-synchronous orbits
  - 20,200 km,
  - 11 hours 58 min
  - Cesium or Rubidium clocks
- At least 4 over each location
- SR: Lose 7,200 ns/day
- GR: Gain 45850 ns/day
- Net, 39 μs/day [or, 500 m/hr]

Triangulation

Equations of intersecting circles:

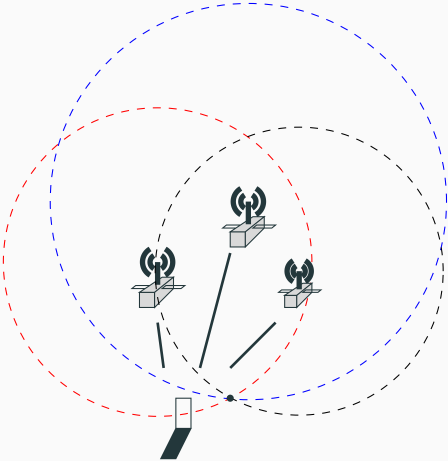
$$\begin{aligned}(x-14)^2+(y-45)^2 &= 39^2.\\(x-80)^2+(y-70)^2 &= 50^2.\\(x-71)^2+(y-50)^2 &= 29^2.\end{aligned}$$

Subtract first and last pairs:

$$\begin{aligned}132x+50y &= 8100,\\18x+40y &= 2100.\end{aligned}$$

Solve: $x = 50, y = 30$.

For satellites, use intersecting spheres and vertical coordinate, z.



Newtonian Gravity and Spacetime

Consider the line element

$$ds^2 = -\left(1+\frac{2\Phi(x^i)}{c^2}\right)(cdt)^2 + \left(1+\frac{2\Phi(x^i)}{c^2}\right)^{-1}(dx^2+dy^2+dz^2).$$

Then, the proper time between points A and B is

$$\begin{aligned}
\tau_{AB} &= \int_A^B d\tau = \int_A^B \left(\frac{ds^2}{c^2}\right)^{1/2} \\
&= \int_A^B \left[\left(1+\frac{2\Phi(x^i)}{c^2}\right)dt^2 - \frac{1}{c^2}\left(1+\frac{2\Phi(x^i)}{c^2}\right)^{-1}(dx^2+dy^2+dz^2)\right]^{1/2} \\
&= \int_A^B dt\left[\left(1+\frac{2\Phi(x^i)}{c^2}\right) - \frac{1}{c^2}\left(1+\frac{2\Phi(x^i)}{c^2}\right)^{-1} v^2\right]^{1/2} \\
&\approx \int_A^B dt\left[1+\frac{2\Phi(x^i)}{c^2} - \frac{1}{c^2}v^2\right]^{1/2} \approx \int_A^B dt\left[1+\frac{1}{c^2}\left(\Phi(x^i)-\frac{1}{2}v^2\right)\right]
\end{aligned}$$

Path of Extremal Proper Time

The proper time between points A and B to first order in $1/c^2$ is

$$\tau_{AB} = \int_A^B dt \left[1 + \frac{1}{c^2}\left(\Phi(x^i) - \frac{1}{2}v^2\right)\right]$$

Extremizing is equivalent to extremizing

$$I = \int_A^B dt \left(\frac{1}{2}v^2 - \Phi(x^i)\right).$$

We have the Lagrangian $L = \frac{1}{2}v^2 - \Phi(x^i)$. The Lagrange equations give

$$\frac{d^2\mathbf{x}}{dt^2} = -\nabla\Phi.$$

Essentially, this is $\mathbf{F} = m\mathbf{a}$.

Sign Conventions

- East Coast (-+++)
  - Minkowski, Einstein, Pauli, Schwinger
  - Spacelike $ds^2 > 0$
  - Minkowski line element $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$.
- West Coast (+ - - -)
  - Bjorken-Drell QFT Text - SLAC
  - Timelike $ds^2 > 0$
  - Minkowski line element $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$.