Homework-#6, Problems: 8, 10, 14 in chapter 4 and 1, 3 in chapter 5

4.8) HEAT SHIELDS

We treat the general case of N heat shields (n = 1, ...,N), in the order $T_\ell \equiv T_0 < T_1 < \ldots < T_N < T_{N+1} \equiv T_u$. The radiative energy flux density between planes n and n-1 is $J_U = \sigma_B(T_n^4 - T_{n-1}^4)$, independent of n. Hence

$$T_n^4 = T_{n-1}^4 + J_U/\sigma_B = T_{n-2}^4 + 2J_U/\sigma_B = T_{n-m}^4 + mJ_U/\sigma_B$$

$$= T_\ell^4 + nJ_{U/\sigma B} \quad . \qquad (S1)$$

Applied to n = N+1 and solved for J_U:

$$J_U = \sigma_B(T_u^4 - T_\ell^4)/(N+1) \quad .$$

Inserted into (S1):

$$T_n^4 = T_\ell^4 + \frac{n}{N+1}(T_u^4 - T_\ell^4) = [(N+1-n)T_\ell^4 + nT_u^4]/(N+1) \quad .$$

For n = N = 1 we have $J_U = \sigma_B(T_u^4 - T_\ell^4)/2$, $T_1^4 = (T_\ell^4 + T_u^4)/2$.

Comment. The instructor might suggest the more general case for extra credit.

4.10) HEAT CAPACITY OF INTERGALACTIC SPACE

For the atoms, $C_H/V = (3/2)nk_B$. For photons, we first re-express the proportionality factor in the expression (20) by inserting the Stefan-Boltzmann constant, $\sigma_B = \pi^2 k_B^4/60\hbar^3 c^2$:

$$U/V = 4\sigma_B T^4/c \quad , \quad C_{rad}/V = 16\sigma_B T^3/c \quad ;$$

$$C_H/C_{rad} = \frac{3ck_B}{32\sigma_B}\,\frac{N/V}{T^3} \cong 2.84\times10^{-10} \quad .$$

Comment. It usually simplifies numerical calculations if one expresses U/V in the form used here, involving σ_B/c and T, rather than the form (20) directly.

4.14) HEAT CAPACITY OF LIQUID ^{4}He AT LOW TEMPERATURES

(a) We need N/V:

$$N/V = \frac{6.02\times10^{23}\text{mol}^{-1} \times 0.145 \text{ g/cm}^3}{4 \text{ g/mol}} = 2.18\times10^{27}\text{atoms/cm}^3 \ .$$

With this, and $v = 2.383\times10^4\text{cm s}^{-1}$, from (44),

$$\theta = (\hbar v/k_B)(6\pi^2 N/V)^{1/3} = 19.83\text{K} \ .$$

(b) For 1 gram, $N = 6.02\times10^{23}/4 \cong 1.5\times10^{23}$. With this and the above value for θ, from (47b), after dividing by 3, to account for the absence of transverse modes,

$$C_V(\text{per gram}) = \frac{1}{3}\,\frac{12\pi^4}{5}\,Nk_B(T/\theta)^3$$

$$= 0.0208 \text{ J g}^{-1}\text{K}^{-4} \times T^3 \ ,$$

which is very close to the experimental value.

Comment. Note that we accounted for the absence of the longitudinal mode by dividing the final heat capacity by 3, not by adjusting the Debye temperature to the lower number of contributing modes.

5.1) CENTRIFUGE

The centrifugal force on a gas particle is $M\omega^2 r$. The corresponding potential energy relative to the axis of rotation is $-M\omega^2 r^2/2$, which serves as external chemical potential. Hence the total chemical potential is

$$\mu = \tau \log[n(r)/n_Q] - M\omega^2 r^2/2 \quad ,$$

which is constant (independent of r) if

$$n(r) = n(0) \exp(M\omega^2 r^2/2\tau) \quad .$$

Note that ωr is the peripheral speed of the centrifuge: The exponential factor is the inverse Boltzmann factor for the kinetic energy of a particle with the velocity v, $\exp(Mv^2/2\tau)$.

Comments. Centrifuging is an effective way to separate isotopes, or to precipitate heavy molecules from solutions. For a gas containing two isotopes with a mass difference ΔM, the heavy-to-light isotope ratio is increased along the perimeter by a factor $\exp(\Delta M v^2/2\tau)$ relative to the center, or conversely, the light-to-heavy isotope ratio is increased by this factor at the center. In such gas centrifuges, peripheral speeds of 350 m s^{-1} have been achieved. For the two uranium isotopes ^{238}U and ^{235}U, with ΔM = 3 amu, this speed leads to a theoretical enrichment factor of 1.076, or 7.6 percent. In this case the gas used is UF_6, and the desired rare isotope ^{235}U accumulates near the axis. For several practical reasons the enrichment factor actually achieved is substantially less than this theoretical ideal.

In the precipitation of heavy organic molecules the mass difference ΔM is the mass different between the molecule and the solvent it displaces. This mass difference can be manipulated over some range, by changing the density of the solvent, by adding a salt, or by similar means.

5.3) POTENTIAL ENERGY OF GAS IN GRAVITATIONAL FIELD

From (18), $n(h) = n(0) \exp(-Mgh/\tau)$, one obtains the total number of atoms in a column, per unit cross-sectional area,

$$N = \int_0^\infty n(h)dh = n(0)\tau/Mg \quad .$$

The total potential energy of the atoms in the column is

$$U_{pot} = \int_0^\infty n(h)\, Mgh\, dh$$

$$= n(0)\tau^2/(Mg) \int_0^\infty e^{-x}\, x\, dx = n(0)\tau^2/Mg = N\tau \quad .$$

The average potential energy per atom is τ. The average kinetic energy is $3\tau/2$, hence the average total energy is $5\tau/2$ and the heat capacity $C = 5/2$.