Homework #4, Problem 2, 3, 5, 7, 8, 10, 14 in chapter 4

4.2) SURFACE TEMPERATURE OF THE SUN

(a) Let J_E denote the radiated energy flux density at the Sun-to-Earth distance D_E from the Sun. Then the total radiated flux is

$$\Phi = 4\pi D_E^2 J_E = 3.845\times10^{26} J\ s^{-1} \quad ,$$

(b) If $R_\odot$ is the radius of the Sun and $J_\odot$ the flux density at that radius,

$$J_\odot = \Phi/4\pi R_\odot^2 = \sigma_B T^4 = 6245\ J\ s^{-1}cm^{-2} \quad ,$$
$$T = (J_\odot/\sigma_B)^{\frac{1}{4}} = 5761\ K \quad .$$

4.3) AVERAGE TEMPERATURE OF THE INTERIOR OF THE SUN

(a) Dimensionally,

$$U \cong -GM_\odot^2/R = -3.77\times10^{48} erg \cong -4\times10^{48} erg \quad . \qquad (S1)$$

If the Sun had a uniform density $\rho = M_\odot/(4\pi R_\odot^3/3)$, the exact result would be

$$U = -G\rho^2 \int_0^\infty (4\pi R^3/3)(4\pi R^2)dR = -3GM_\odot^2/5R_\odot \quad .$$

We continue with the value (S1).

(b) The Sun consists mostly of hydrogen atoms; their number is

$$N \cong M_\odot/M_H = 1.195\times10^{57} \cong 1\times10^{57} \quad .$$

From $3\tau N/2 = -U/2$:

$$T = -U/3k_B N = 9.66\times10^6 K \cong 10^7 K \quad .$$

4.5) SURFACE TEMPERATURE OF THE EARTH

The solar energy flux density at the Earth's orbit is $J_E = \sigma_B T_\odot^4 \times (R_\odot/D_E)^2$. The flux intercepted by the earth is

$$\Phi = \pi R_E^2 J_E = \pi \sigma_B T_\odot^4 \times (R_E R_\odot/D_E)^2 \quad .$$

The flux re-radiated by the Earth is

$$\Phi = \sigma_B T_E^4 \times 4\pi R_E^2 \quad .$$

Equating the two fluxes leads to

$$T_E = (R_\odot/2D_E)^{1/2} T_\odot \cong 280 \text{ K} \quad .$$

Note that the radius of the Earth drops out.

4.7) FREE ENERGY OF A PHOTON GAS

(a) The partition function for a single mode with frequency ω_n is, from (3),

$$Z_n = [1 - \exp(-\hbar\omega_n/\tau)]^{-1} \quad . \tag{S1}$$

The different modes are independent of each other; therefore the partition function for the overall photon gas is simply the product of all single-mode partition functions:

$$Z = \prod_n Z_n \quad .$$

Because of (S1) this is the same as (53).

(b) $F = -\tau \log Z = -\tau \quad \log Z_n$. With (S1) this is the same as (54). Transformation to an integral:

$$\sum_n \ldots = \pi \int_0^\infty \ldots\ n^2 dn = \pi(\tau L/\pi\hbar c)^3 \int_0^\infty \ldots\ u^2 du \quad ,$$

with $u = (\pi\hbar c/\tau L)n$; see Problem 4.1. Integration by parts:

$$\int_0^\infty \log(1-e^{-u})u^2du = \frac{u^3}{3}\log(1-e^{-u})\Big|_0^\infty - \frac{1}{3}\int_0^\infty \frac{u^3du}{e^u-1} = -\frac{\pi^4}{45} \quad .$$

The integrated term vanishes at both $u = 0$ and $u = \infty$, and the integral is the same as in (19), $\pi^4/15$. Inserted into F:

$$V = \tau \times \pi(\tau L/\pi\hbar c)^3 \times (-\pi^4/45) = -\pi^2 V\tau^4/45c^3\hbar^3 \quad .$$

4.8) HEAT SHIELDS

We treat the general case of N heat shields ($n = 1, \ldots, N$), in the order $T_\ell \equiv T_0 < T_1 < \ldots < T_N < T_{N+1} \equiv T_u$. The radiative energy flux density between planes n and n-1 is $J_U = \sigma_B(T_n^4 - T_{n-1}^4)$, independent of n. Hence

$$T_n^4 = T_{n-1}^4 + J_U/\sigma_B = T_{n-2}^4 + 2J_U/\sigma_B = T_{n-m}^4 + mJ_U/\sigma_B$$

$$= T_\ell^4 + nJ_{U/\sigma B} \quad . \qquad \text{(S1)}$$

Applied to $n = N+1$ and solved for J_U:

$$J_U = \sigma_B(T_u^4 - T_\ell^4)/(N+1) \quad .$$

Inserted into (S1):

$$T_n^4 = T_\ell^4 + \frac{n}{N+1}(T_u^4 - T_\ell^4) = [(N+1-n)T_\ell^4 + nT_u^4]/(N+1) \quad .$$

For $n = N = 1$ we have $J_U = \sigma_B(T_u^4 - T_\ell^4)/2$, $T_1^4 = (T_\ell^4 + T_u^4)/2$.

4.10) HEAT CAPACITY OF INTERGALACTIC SPACE

For the atoms, $C_H/V = (3/2)nk_B$. For photons, we first re-express the proportionality factor in the expression (20) by inserting the Stefan-Boltzmann constant, $\sigma_B = \pi^2 k_B^4/60\hbar^3c^2$:

$$U/V = 4\sigma_B T^4/c \quad , \quad C_{rad}/V = 16\sigma_B T^3/c \quad ;$$

$$C_H/C_{rad} = \frac{3ck_B}{32\sigma_B}\frac{N/V}{T^3} \cong 2.84\times10^{-10} \quad .$$

4.14) HEAT CAPACITY OF LIQUID ^{4}He AT LOW TEMPERATURES

(a) We need N/V:

$$N/V = \frac{6.02\times10^{23}\text{mol}^{-1} \times 0.145\ \text{g/cm}^3}{4\ \text{g/mol}} = 2.18\times10^{27}\text{atoms/cm}^3 \ .$$

With this, and $v = 2.383\times10^4 \text{cm s}^{-1}$, from (44),

$$\theta = (\hbar v/k_B)(6\pi^2 N/V)^{1/3} = 19.83\text{K} \ .$$

(b) For 1 gram, $N = 6.02\times10^{23}/4 \cong 1.5\times10^{23}$. With this and the above value for θ, from (47b), after dividing by 3, to account for the absence of transverse modes,

$$C_V(\text{per gram}) = \frac{1}{3}\,\frac{12\pi^4}{5}\,Nk_B(T/\theta)^3$$
$$= 0.0208\ \text{J g}^{-1}\text{K}^{-4} \times T^3 \ ,$$

which is very close to the experimental value.