Homework#3, problem 1, 2, 3, 10, 11 in chapter 3

3.1) FREE ENERGY FOR TWO STATE SYSTEM

(a) By inserting (13) into (55):

$$F = -\tau \log Z = -\tau \log[1+\exp(-\varepsilon/\tau)] \quad .$$

(b) Inserted into (49),

$$\sigma = -(\partial F/\partial\tau) = \log[1+\exp(-\varepsilon/\tau)] + (\varepsilon/\tau)/[1+\exp(\varepsilon/\tau)], \qquad (S1)$$

$$U = F + \tau\sigma = \varepsilon/[1+\exp(\varepsilon/\tau)] \quad ,$$

the same as (14) obtained directly from Z.

Comment. As $\tau \to 0$ and hence $\exp(-\varepsilon/\tau) \to 0$, the logarithm in (S1) may be expanded: $\log[1+\exp(-\varepsilon/\tau)] \to \exp(-\varepsilon/\tau)$. Then $\sigma \to (1+\varepsilon/\tau)\exp(-\varepsilon/\tau) \to 0$. The exponential factor goes to zero faster than any inverse power of τ goes to infinity: Both σ and all derivatives of σ vanish when $\tau \to 0$, as shown in Fig. 11. The high-temperature limit of σ is obtained by letting $\exp(-\varepsilon/\tau) \to 1$, in (S1): $\sigma \to \log 2 + \varepsilon/2\tau \to \log 2$.

3.2) MAGNETIC SUSCEPTIBILITY

(a) For a single magnet, with $\varepsilon = \mp mB$:

$$Z_1 = \exp(mB/\tau) + \exp(-mB/\tau) = 2\cosh(mB/\tau) \quad , \qquad (S1)$$

$$\langle m\rangle = [m\exp(+mB/\tau) - m\exp(-mB/\tau)]/Z_1 = m\tanh(mB/\tau).$$

The magnetization M is obtained by multiplying by the particle concentration n:

$$M = n\langle m\rangle = nm\tanh(mB/\tau) \quad .$$

For weak fields, $mB \ll \tau$:

$$M = nm^2B/\tau \quad . \qquad (S2)$$

For strong fields, $mB \gg \tau$:

$$M = nm.$$

These two limits are seen in Fig. 3.12.

(b) Inserting (S1) into (55) and multiplying by n:

$$F = -n\tau \log Z_1 = -n\tau \log[2 \cosh(mB/\tau)] \quad .$$

To express $\cosh(mB/\tau)$ as a function of $x = M/nm = \tanh(mB/\tau) = \tanh y$ we use the relation

$$1/\cosh^2 y = (\cosh^2 y - \sinh^2 y)/\cosh^2 y = 1 - \tanh^2 y = 1 - x^2.$$

We next write

$$\log[2 \cosh y] = -\tfrac{1}{2} \log(1/4\cosh^2 y) = -\tfrac{1}{2} \log[(1-x^2)/4].$$

With this:

$$F = +(n\tau/2) \log[(1-x^2)/4] \quad .$$

(c) The susceptibility is defined as

$$\chi = dM/dB.$$

In the limit $mB \ll \tau$, from (S2):

$$\chi = nm^2/\tau \quad .$$

3.3) <u>FREE ENERGY OF AN HARMONIC OSCILLATOR</u>

The partition function is a geometric series:

$$Z = \sum_{s=0}^{\infty} \exp(-s\hbar\omega/\tau) = 1/[1-\exp(-\hbar\omega/\tau)].$$

(a) Inserted into (55):

$$F = -\tau \log Z = \tau \log[1-\exp(-\hbar\omega/\tau)] \quad . \tag{87}$$

At high temperatures, $\hbar\omega/\tau \ll 1$, so that $1-\exp(-\hbar\omega/\tau) \cong \hbar\omega/\tau$. Hence from (S1):

$$F \cong \tau \log(\hbar\omega/\tau) \quad . \tag{S2}$$

(b) The expression (88) follows directly by inserting (87) into (49), $\sigma = -(\partial F/\partial\tau)$.

Comment. The low-temperature ($\tau \ll \hbar\omega$) behavior of the harmonic oscillator is the same as for the two state system with $\varepsilon = \hbar\omega$, as is apparent from comparing Figs. 3.13 and 3.14 with Figs. 3.11 and 3.4: Only the two lowest states matter. The high-temperature behavior ($\tau \gg \hbar\omega$) is quite different, because the number of accessible states is not limited to 2. In this limit, from (S2):

$$\sigma = -(\partial F/\partial\tau) \rightarrow 1 + \log(\tau/\hbar\omega).$$

If this is inserted into (17a):

$$C_V = \tau(\partial\sigma/\partial\tau) \rightarrow 1,$$

in fundamental units.

3.10) ELASTICITY OF POLYMERS

(a) The problem is formally almost the same as the model spin system of chapter 1. Suppose that the molecular structure of each link is not invariant under inversion (example: ABC ≠ CBA). We may then associate a vector with each link, pointing either to the left or to the right, analogously to the spin vector pointing up or down. If $N_{\rightarrow}$ and $N_{\leftarrow}$ are the numbers of links with vectors pointing to the left and to the right, we may define a "length excess" 2s analogously to the spin excess of (1.11): $2s = N_{\rightarrow} + N_{\leftarrow}$. The number of states with a given length excess is the same as g(N,s) in (1.15):

$$g(N,s) = N!/[(\tfrac{1}{2}N+s)!(\tfrac{1}{2}N-s)!] \quad , \quad N = N_{\rightarrow} + N_{\leftarrow} \quad . \qquad \text{(S1)}$$

The value of g(N,s) does not change when the sign of s changes. The number of states with a given magnitude of s is obtained by adding g(N,s) + g(N,-s), which yields (97).

(b) By inserting (1.36) into (S1), and by setting $s^2 = \ell^2/4\rho^2$:

$$\begin{aligned} g(N,s) + g(N,-s) &= 2g(N,0)\exp(-2s^2/N) \quad , \\ \sigma(\ell) = \log[g(N,s) + g(N,-s)] &= \log[2g(N,0)] - 2s^2/N \\ &= \log[g(N,0)] - \ell^2/2N\rho^2 \quad . \qquad (98) \end{aligned}$$

(c) $(\partial\sigma/\partial\ell)_N = -\ \ell/N\rho^2 \quad ,$

$$f = -\ \tau(\partial\sigma/\partial\ell)_N = +\ \ell\tau/N\rho^2 \quad . \qquad (99)$$

3.11) ONE-DIMENSIONAL GAS

In one dimension, the orbital energies are, from (58) and (59), $\varepsilon_n = \varepsilon_1 n^2$, where $\varepsilon_1 = (\hbar^2/2M)(\pi/L)^2$ and n is a positive integer. The single-particle partition function is

$$Z_1 = \sum_n \exp(-\varepsilon_1 n^2/\tau) \cong \int_0^\infty \exp(-\varepsilon_1 n^2/\tau)dn$$

$$= (\pi\tau/4\varepsilon_1)^{\frac{1}{2}} = n_{Q1}L \quad ,$$

where

$$n_{Q1} = (M\tau/2\pi\hbar^2)^{\frac{1}{2}} = (n_Q)^{1/3}$$

is the one-dimensional quantum concentration analogous to the three-dimensional quantum concentration n_Q defined in (62) and (63).

For N particles: $Z_N = Z_1^N/N!$,

$$F = -\tau \log Z_N = \tau \log N! - \tau N \log Z_1$$

$$\cong \tau(N \log N - N) - \tau N \log(n_{Q1}L)$$

$$= \tau N[\log(n/n_{Q1}) - 1],$$

where $n = N/L$. With the help of $\partial \log n_{Q1}/\partial\tau = 1/2\tau$:

$$\sigma = -(\partial F/\partial\tau)_n = N[\log(n/n_Q) - 3/2],$$

which should be compared with (76).