Example 3.1. The spin 1/2 problem is mathematically identical to the random walk (or "drunkard's walk") problem in one dimension. The random walk can be visualized as follows. A drunk starts from a lamp post and takes steps of length ℓ to the left or to the right at random with equal probability, using the kerb as a guide, so that all his steps fall on a line. What is the probability that after N steps, he will (a) be back at the lamp post, or (b) be at a distance $n\ell$ from it?

(a) at the lamp post $2S=0 \rightarrow \therefore S=0$

$$g = \frac{N!}{(\frac{N}{2}+S)!\,(\frac{N}{2}-S)!} = \frac{N!}{(\frac{N}{2})!\,(\frac{N}{2})!} \qquad P(S=0) = \frac{g}{2^N}$$

(b) at a distance $n\ell$ from the lamp post. $2S = \pm n$.

For $S = \frac{n}{2}$ $\quad g(S=\frac{n}{2}) = \dfrac{N!}{(\frac{N}{2}+\frac{n}{2})!\,(\frac{N}{2}-\frac{n}{2})!}$

$S = -\frac{n}{2}$ $\quad g(S=-\frac{n}{2}) = \dfrac{N!}{(\frac{N}{2}-\frac{n}{2})!\,(\frac{N}{2}+\frac{n}{2})!}$

$$P(S=\pm\frac{n}{2}) = \frac{2N!}{(\frac{N+n}{2})!\,(\frac{N-n}{2})!} \Big/ 2^N$$

2.1) ENTROPY AND TEMPERATURE

(a) $\sigma = \log g = \log C + (3N/2)\log U.$

$(\partial\sigma/\partial U)_N = 1/\tau = (3N/2)(1/U)$, hence $U = 3N\tau/2$.

(b) $(\partial^2\sigma/\partial U^2)N = (3N/2)(-1/U^2) < 0.$

2.2) PARAMAGNETISM

States with spin excess 2s have the energy $U = -2smB$, hence $s = -U/2mB$. Insertion into (40) yields (41). From (41) obtain $(\partial\sigma/\partial U)_{N,B} = 1/\tau = -U/m^2B^2N$. Solve for U: $U = \langle U\rangle = -m^2B^2N/\tau = -MB$. Hence $M/Nm = mB/\tau$.

2.3) QUANTUM HARMONIC OSCILLATOR

(a) From (1.55):

$$\sigma = \log g = \log(N+n-1)! - \log n! - \log(N-1)!$$

$$\cong \log(N+n)! - \log n! - \log N!$$

Replacing N-1 by N is equivalent to neglecting a term $-\log[(N+n)/N]$, which for large N is small compared to the factorial terms. With the Stirling approximation:

$$\sigma \cong (N+n)\log(N+n) - n\log n - N\log N$$

$$= (N+n)\log[(N+n)/N] - n\log(n/N)$$

$$= N[(1+n/N)\log(1+n/N) - (n/N)\log(n/N)].$$

(b) Set $n = U/\hbar\omega$ and $N\hbar\omega = U_0(N)$:

$$\sigma(U,N) = N[(1+U/U_0)\log(1+U/U_0) - (U/U_0)\log(U/U_0)].$$

With the abbreviation $x = U/U_0$:

$$1/\tau = (\partial\sigma/\partial U)_N = (1/U_0)(\partial\sigma/\partial x)_N = (N/U_0)[\log(1+x)-\log x]$$
$$= (1/\hbar\omega) \log(1+U_0/U).$$

Solving for U/N yields $U/N = \hbar\omega/[\exp(\hbar\omega/\tau) - 1]$.

Alternate method: For large n:

$$d(\log n!)/dn \cong \log n! - \log(n-1)! = \log n.$$

With this,

$$1/\tau = (1/\hbar\omega)(\partial\sigma/\partial n)_N = (1/\hbar\omega)[\log(N+n) - \log n]$$
$$= (1/\hbar\omega) \log(1+N/n),$$

which is the equivalent to the result above.

2.4) THE MEANING OF NEVER

(a) The probability a correct key will be struck is 1/44. The probability a sequence of 10^5 keys will be correct is

$$(1/44)^{100,000} = 10^{-164,345}.$$

(b) On a single typewriter the number of keys that can be struck in 10^{18}s at 10 keys/s is 10^{19}. The Hamlet sequence may start with any of these except the last 10^5-1. Thus there are $10^{19} - 10^5 + 1 \cong 10^{19}$ possible starts. The probability of a correct Hamlet sequence on one typewriter is

$$10^{19}\times10^{-164,345} = 10^{-164,326}.$$

For 10^{10} monkeys:

$$10^{10}\times10^{-164,326} = 10^{-164,316}.$$