Homework #5. Problems: 1, 2, 3, 5, 7 in chapter 4

4.1) NUMBER OF THERMAL PHOTONS

We need

$$\sum_n \langle s_n \rangle = \sum_n 1/[\exp(\hbar\omega_n/\tau) - 1] = \sum_n 1/(e^u - 1) \quad ,$$

where $\omega_n = n\pi c/L$, $u = (\pi\hbar c/\tau L)n$. The sum is converted to an integral, as in (17), but with an additional factor 2 for the two independent polarizations:

$$\sum_n \ldots = \pi \int_0^\infty \ldots\ n^2 dn = \pi(\tau L/\pi\hbar c)^3 \int_0^\infty \ldots\ u^2 du \quad .$$

The factor preceding the final integral may be written $\pi^{-2}V(\tau/\hbar c)^3$, as in (48). The integral has the value, evaluated below,

$$\int_0^\infty \frac{u^2 du}{e^u - 1} = 2.404113806 \quad .$$

Comment. Integrals of the form

$$I = \int_0^\infty \frac{u^n du}{e^u - 1} \quad \text{with } n \geq 0$$

occur often in thermal physics. They may be evaluated by expanding the denominator into a (converging) geometric series:

$$I_n = \sum_{m=1}^\infty \int_0^\infty e^{-mu} u^n du = \sum_{m=1}^\infty \frac{1}{m^{n+1}} \int_0^\infty e^{-v} v^n dv = n! \sum_{m=1}^\infty \frac{1}{m^{n+1}} \ .$$

For odd integer values of n the sums have simple closed-form values ($I_1 = \pi^2/6$, $I_3 = \pi^2/15$), listed in many tables. For other values no such simple relations exist. Purely numerical values for integer n are given in many of the more elaborate tables, but the series are readily (although slowly) summed on a programmable calculator. The calculation can be greatly speeded up by summing only a relatively small number of terms and approximating the remainder as an integral:

$$\sum_{m=1}^{\infty} \frac{1}{m^{n+1}} \cong \sum_{m=1}^{M} \frac{1}{m^{n+1}} + \int_{M+\frac{1}{2}}^{\infty} \frac{dm}{m^{n+1}} \quad .$$

The integral has the value $1/[n(M+\frac{1}{2})^n]$. For $n = 2$, $M = 10$ one obtains in this way $I_2 = 2.404134$, a result within less than 1 part in 10^5 of the exact value, 2.404114. The accuracy increases with increasing M.

4.2) SURFACE TEMPERATURE OF THE SUN

(a) Let J_E denote the radiated energy flux density at the Sun-to-Earth distance D_E from the Sun. Then the total radiated flux is

$$\Phi = 4\pi D_E^2 J_E = 3.845\times10^{26} \text{J s}^{-1} \quad ,$$

(b) If $R_\odot$ is the radius of the Sun and $J_\odot$ the flux density at that radius,

$$J_\odot = \Phi/4\pi R_\odot^2 = \sigma_B T^4 = 6245 \text{ J s}^{-1}\text{cm}^{-2} \quad ,$$

$$T = (J_\odot/\sigma_B)^{\frac{1}{4}} = 5761 \text{ K} \quad .$$

4.3) AVERAGE TEMPERATURE OF THE INTERIOR OF THE SUN

(a) Dimensionally,

$$U \cong - GM_\odot^2/R = - 3.77\times10^{48}\text{erg} \cong - 4\times10^{48}\text{erg} \quad . \qquad (S1)$$

If the Sun had a uniform density $\rho = M_\odot/(4\pi R_\odot^3/3)$, the exact result would be

$$U = - G\rho^2 \int_0^\infty (4\pi R^3/3)(4\pi R^2)dR = - 3GM_\odot^2/5R_\odot \quad .$$

We continue with the value (S1).

(b) The Sun consists mostly of hydrogen atoms; their number is

$$N \cong M_\odot/M_H = 1.195\times10^{57} \cong 1\times10^{57} \quad .$$

From $3\tau N/2 = -U/2$:

$$T = - U/3k_BN = 9.66\times10^6\text{K} \cong 10^7\text{K} \quad .$$

4.5) SURFACE TEMPERATURE OF THE EARTH

The solar energy flux density at the Earth's orbit is $J_E = \sigma_B T_\odot^4 \times (R_\odot/D_E)^2$. The flux intercepted by the earth is

$$\Phi = \pi R_E^2 J_E = \pi\sigma_B T_\odot^4 \times (R_E R_\odot/D_E)^2 \quad .$$

The flux re-radiated by the Earth is

$$\Phi = \sigma_B T_E^4 \times 4\pi R_E^2 \quad .$$

Equating the two fluxes leads to

$$T_E = (R_\odot/2D_E)^{\frac{1}{2}} T_\odot \cong 280\ \mathrm{K} \quad .$$

Note that the radius of the Earth drops out.

Comment. The problem lends itself to various elaborations, such as estimating the temperature of the solar energy panels of a satellite, as a function of the attitude of the panels, and of the emissivity of the back of the panel.

$$U = -\,G\rho^2 \int_0^\infty (4\pi R^3/3)(4\pi R^2)\,dR = -\,3GM_\odot^2/5R_\odot \quad .$$

We continue with the value (S1).

(b) The Sun consists mostly of hydrogen atoms; their number is

$$N \cong M_\odot/M_H = 1.195\times10^{57} \cong 1\times10^{57} \quad .$$

From $3\tau N/2 = -U/2$:

$$T = -\,U/3k_B N = 9.66\times10^{6}\mathrm{K} \cong 10^{7}\mathrm{K} \quad .$$

4.7) FREE ENERGY OF A PHOTON GAS

(a) The partition function for a single mode with frequency ω_n is, from (3),

$$Z_n = [1 - \exp(-\hbar\omega_n/\tau)]^{-1} \quad . \qquad \text{(S1)}$$

The different modes are independent of each other; therefore the partition function for the overall photon gas is simply the product of all single-mode partition functions:

$$Z = \prod_n Z_n \quad .$$

Because of (S1) this is the same as (53).

(b) $F = -\tau \log Z = -\tau \quad \log Z_n$. With (S1) this is the same as (54). Transformation to an integral:

$$\sum_n \ldots = \pi \int_0^\infty \ldots n^2 dn = \pi(\tau L/\pi\hbar c)^3 \int_0^\infty \ldots u^2 du \quad ,$$

with $u = (\pi\hbar c/\tau L)n$; see Problem 4.1. Integration by parts:

$$\int_0^\infty \log(1-e^{-u})u^2 du = \frac{u^3}{3} \log(1-e^{-u}) \Big|_0^\infty - \frac{1}{3}\int_0^\infty \frac{u^3 du}{e^u - 1} = -\frac{\pi^4}{45} \quad .$$

The integrated term vanishes at both $u = 0$ and $u = \infty$, and the integral is the same as in (19), $\pi^4/15$. Inserted into F:

$$V = \tau \times \pi(\tau L/\pi\hbar c)^3 \times (-\pi^4/45) = -\pi^2 V\tau^4/45c^3\hbar^3 \quad .$$