Homework # 12, problems: 4, 6, 8, 9, 10 in chapter 8

8.4) HEAT ENGINE-REFRIGERATOR CASCADE

Taking the last question first, the answer to it is clearly: No. The cascade is simply another more complicated heat engine in its own right, which is subject to the Carnot limit just as the individual heat engine and refrigerator are.

The heat engine operates between the temperatures τ_h and τ_r. It is instructive to go beyond the problem statement and to include irreversibility. In this case, from (9) and (10), with the substitution $\ell \to r$,

$$Q_r \geqq (\tau_r/\tau_h)Q_h \quad , \tag{S1}$$

$$W_e = Q_h - Q_r \leqq (1-\tau_r/\tau_h)Q_h \quad , \tag{S2}$$

where the equals sign holds in the reversible limit.

For the refrigerator, τ_ℓ is actually the upper temperature and τ_r the lower temperature. If we substitute first $\ell \to r$, then $h \to \ell$ in (15) and (16) we obtain

$$Q_\ell \geqq (\tau_\ell/\tau_r)Q_r,$$

$$W_r = Q_\ell - Q_r \geqq (\tau_\ell/\tau_r - 1)Q_r \geqq (\tau_\ell/\tau_h - \tau_r/\tau_h)Q_h, \tag{S3}$$

where in the last inequality we utilized (S1). The work generated by the cascade is, with (S2) and (S3),

$$W = W_e - W_r \leqq (1-\tau_r/\tau_h)Q_h - (\tau_\ell/\tau_h - \tau_r/\tau_h)Q_h = (1-\tau_\ell/\tau_h)Q_h \quad ,$$

which is the expected overall Carnot relation equivalent to (11) for the cascade.

8.6) <u>ROOM AIR CONDITIONER</u>

(a) We have $dQ_\ell/dt = A\times(T_h - T_\ell)$. From this and (12):

$$P = dW/dt = [(T_h - T_\ell)/T_\ell]\, dQ_\ell/dt = A\times(T_h - T_\ell)^2/T_\ell \quad . \tag{S1}$$

This may be re-arranged into a quadratic equation for T_ℓ:

$$T_\ell^2 - (2T_h + P/A)T_\ell + T_h^2 = 0 \quad .$$

Its relevant solution ($T_\ell < T_h$) is

$$T_\ell = (T_h + P/2A) - [(T_h + P/2A)^2 - T_h^2]^{1/2} \quad .$$

(b) From (S1):

$$A = PT_\ell/(T_h - T_\ell)^2 = 2 \text{ kW} \times 290 \text{ K}/(20 \text{ K})^2 = 1.45 \text{ kW K}^{-1} \quad .$$

8.8) <u>GEOTHERMAL ENERGY</u>

The work $đW$ extractable from the heat $đQ_h$ by a Carnot engine is, from (10),

$$đW = [(T_h - T_\ell)/T_h]đQ_h = -MC(1 - T_\ell/T_h)dT_h \quad ,$$

where T_h is the temperature at which $đQ_h$ is removed from the rock. The total work extractable follows by integration over T_h:

$$W = -MC\int_{T_i}^{T_f}(1-T_\ell/T_h)dT_h = -MC[T_f-T_i-T_\ell\log(T_f/T_i)] \ .$$

Numerically, $MC = 10^{17}J\ K^{-1}$. The value of the brackets is -249 K. Witn these numbers:

$$W = 10^{17}J\ K^{-1} \times 249\ K/(3.6\times10^{6}J/kWh) = 6.92\times10^{12}kWh \ .$$

8.9) COOLING OF NON-METALLIC SOLID TO T = 0.

Let $đQ_\ell = -CdT_\ell = -aT_\ell^{\,3}dT_\ell$ be the heat removed from the solid in lowering its temperature from T_ℓ to $T_\ell + dT_\ell$ (we assume $dT_\ell<0$). The reversible work to do so is, from (12):

$$đW = [(T_h-T_\ell)/T_\ell]dQ_\ell = -a[T_hT_\ell^{\,2}-T_\ell^{\,3}]dT_\ell \ .$$

The total work required to cool the solid from $T_\ell = T_h$ to $T_\ell = 0$ follows be integration

$$W = -a\int_{T_h}^{0}[T_hT_\ell^{\,2} - T_\ell^{\,3}]dT_\ell$$

$$= a\left[\tfrac{1}{3}T_hT_\ell^{\,3} - \tfrac{1}{4}T_\ell^{\,4}\right]_0^{T_h} = \frac{1}{12}\,aT_h^{\,4} \ .$$

8.10) IRREVERSIBLE EXPANSION OF A FERMI GAS

The average energy per particle remains unchanged. Initially, from (7.10) and (7.7):

$$\langle\varepsilon\rangle = 3\varepsilon_F/5 = (3\hbar^2/10M)(3\pi^2 N/V_i)^{2/3} \quad .$$

After expansion to a concentration $n \ll 2n_Q$ the gas behaves like an ideal gas, with $\langle\varepsilon\rangle = 3\tau/2$. Equating the two values:

$$\tau = 2\varepsilon_F/5 = (2\hbar^2/10M)(3\pi^2 N/V\ )^{2/3} \quad . \qquad \text{(S1)}$$

The condition $n \ll 2n_Q$ can be written, with (3.63):

$$N/V_f \ll 2n_Q = 2(M\tau/2\pi\hbar^2)^{3/2} \quad .$$

Inserting τ from (S1):

$$N/V_f \ll (3/5)(\pi/10)^{1/2}(N/V_i) \text{ or } V_f/V_i \gg (5/3)(10/\pi)^{1/2} \cong 3 \quad .$$

Thus any expansion by a factor large compared to 3 will do.

(a) 7860 K. (b) 9.22×10^5K.