Homework-10, Problems: 3, 5, 7, 11, 12, 14 in chapter 7

7.3) PRESSURE AND ENTROPY OF DEGENERATE FERMI GAS

(a) From (10) and (7):

$$U_0 = \frac{3}{5}N\varepsilon_F, \text{ where } \varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3} \quad . \tag{S1}$$

The pressure is obtained from (3.26):

$$p = -(\partial U/\partial V)_{\sigma,N} = -(\partial U_0/\partial V)_N \quad .$$

In the second equality we have set $U = U_0$ and have dropped the subscript σ, because a volume change on the system while the system is kept in its ground state is a change at constant entropy. Because, from (S1), $U_0 \propto V^{-2/3}$, we have

$$p = \frac{2}{3}\frac{U_0}{V} = \frac{2}{5}\frac{N}{V}\varepsilon_F \quad .$$

Insertion of the expression for ε_F yields (90), but the form given here is more "physical": The pressure is a fixed multiple of the product of particle concentration and Fermi energy.

(b) The fastest way to obtain σ is with the help of the relation (3.17a),

$$C_V = \tau(\partial\sigma/\partial\tau)_V \quad , \tag{S2}$$

from the low-temperature heat capacity of the Fermi gas, already calculated in (37). According to (37), C_V is simply proportional to τ. Because of (S2), together with the requirement that $\sigma \to 0$ as $\tau \to 0$, this implies that σ is equal to C_V:

$$\sigma = C_V = \pi^2 N\tau/2\varepsilon_F \quad . \tag{S3}$$

An example of an alternate way to obtain σ is the following. From $\sigma = -\partial F/\partial\tau$ and $\mu = \partial F/\partial N$ follow the Maxwell relation

$$\left(\frac{\partial\sigma}{\partial N}\right)_\tau = -\left(\frac{\partial\mu}{\partial\tau}\right)_N \quad . \tag{S4}$$

The quantity μ is calculated in the next problem, see (7.4.S4). If (7.4.S4) is inserted into (S3):

$$\left(\frac{\partial\sigma}{\partial N}\right)_\tau = \pi^2\tau/6\varepsilon_F \quad . \tag{S5}$$

From (7) follows $1/\varepsilon_F \propto N^{-2/3}$. Integrating this is the same as multiplying by 3N,

$$\int N^{-2/3} dN = 3N^{1/3} = 3N\times N^{-2/3} \quad .$$

Combining this with (S5) leads again to (S3).

7.5) LIQUID ^{3}He AS A FERMI GAS

(a) With $M(^3He) \cong 3$ amu $\cong 4.98\times10^{-24}$g we have $N/V = 0.081$ g cm^{-3}/3 amu $\cong 1.63\times10^{22}$cm^{-3}. Next, from (40):

$$\varepsilon_F = (\hbar^2/2m)(3\pi^2N/V)^{2/3} = 6.86\times10^{-16}\text{erg} \quad ,$$

$$T_F = \varepsilon_F/k_B = 4.97 \text{ K} \quad ,$$

$$v_F = (2\varepsilon_F/M)^{1/2} = 1.66\times10^4 \text{cm s}^{-1} \quad .$$

(b) From (38),

$$C_V = (\pi^2/2)N_Ak_BT/T_F = 0.99 \text{ K} \quad ,$$

which is only 1/(2.91) of the observed value. The motion of individual ^{3}He atoms in liquid ^{3}He is correlated. To move a ^{3}He atom requires pushing other atoms out of the way. The kinetic energy of the other atoms getting out of the way is part of the kinetic energy involved in moving the atom, making it appear as if each ^{3}He atom had an effective mass greater than its real mass. The heat capacity suggests an effective mass 2.91 times $M(^3He)$.

7.7) PHOTON CONDENSATION

We solve (4.48) for $\tau^3 = (k_BT_C)^3$:

$\tau^3 = (\hbar c)^3\pi^2(N/V)/2.404$; $T_C = (\hbar c/k_B)\times(\pi^2N/2.404V)^{1/3}$.

For $N/V = 10^{20}$cm^{-3} we have $T_C = 1.70\times10^6$K.

7.8) ENERGY, HEAT CAPACITY, AND ENTROPY OF DEGENERATE BOSON GAS

Only the uncondensed bosons contribute to the energy. Their number $N_e(\tau)$ was calculated in (68); the expression for their energy differs from (68) only by an additional factor $\varepsilon = \tau x$:

$$U(\tau) = \frac{V}{4\pi^2}\left(\frac{2M}{\hbar^2}\right)^{3/2} \tau^{5/2}\int_0^\infty \frac{x^{3/2}dx}{e^x-1} = AN_e(\tau)\tau \quad , \qquad (S1)$$

where A is the ratio of the integral in (S1) to the integral (69). Numerically, A = 0.770 (see below). We use (73) to write $U(\tau)$ in the final form

$$U(\tau) = AN\tau^{5/2}\tau_E^{-3/2} \quad .$$

From this:

$$C_V(\tau) = (\partial U/\partial\tau)_{V,N} = \frac{5}{2}\,AN(\tau/\tau_E)^{3/2} \quad .$$

Because of (3.17), $C_V = \tau(\partial\sigma/\partial\tau)$:

$$(\partial\sigma/\partial\tau)_{V,N} = \frac{5}{2}\,AN\tau^{1/2}\tau_E^{-3/2} \;,$$

$$\sigma(\tau) = \frac{5}{3}\,AN(\tau/\tau_E)^{3/2} \quad ,$$

where in the last line we have used $\sigma(0) = 0$.

<u>Comment</u>. The integral in (S1) may be evaluated by the method in the footnote on p.204 of the text:

$$\int_0^\infty \frac{x^{3/2}dx}{e^x-1} = \left(\sum_{s=1}^\infty s^{-5/2}\right)\int_0^\infty y^{3/2}e^{-y}dy \quad .$$

The infinite sum is easily summed to 1.341 on a programmable calculator. The integral is a gamma function (see Appendix A): $\Gamma(5/2) = 3/2\,\Gamma(3/2)$, where $\Gamma(3/2)$ is the integral in the footnote on p.204. Hence

$$A = (3/2)\times 1.341/2.612 = 0.7702 \quad .$$

7.11) FLUCTUATIONS IN A FERMI GAS

By definition, $\langle(\Delta N)^2\rangle = \langle(N-\langle N\rangle)^2\rangle = \langle N^2\rangle - \langle N\rangle^2$. For fermions, only N = 0,1 are allowed. For those values, $N^2 = N$ and $\langle N^2\rangle = \langle N\rangle$. Therefore:

$$\langle(\Delta N)^2\rangle = \langle N\rangle - \langle N\rangle^2 = \langle N\rangle(1 - \langle N\rangle) \quad .$$

This result may also be obtained less elegantly but perhaps more systematically from (5.83) -- see solution to Problem 5.20 -- by inserting (6.4) for $\langle N\rangle = f(\varepsilon)$.

7.12) FLUCTUATION IN A BOSE GAS

The trick $\langle N^2\rangle = \langle N\rangle$ of the preceding problem is not applicable to bosons. We draw on (5.83) -- see solution to Problem 5.10 -- by inserting (6.10) for $\langle N\rangle = f(\varepsilon)$:

$$\langle(\Delta N)^2\rangle = \tau\ \partial\langle N\rangle/\partial\mu = \tau \frac{\partial}{\partial\mu} \frac{1}{\exp[(\varepsilon-\mu)/\tau]-1} \quad .$$

$$= \frac{\exp[(\varepsilon-\mu)/\tau]}{\{\exp[(\varepsilon-\mu)/\tau]-1\}^2} = \langle N\rangle(1+\langle N\rangle) \quad .$$

7.14) <u>TWO ORBITAL BOSON SYSTEM</u>

From (53) and (54), or from (6.10):

$$\exp[(\varepsilon-\mu)/\tau] = 1 + 1/f(\varepsilon) \quad , \tag{S1}$$

$$\exp(-\mu/\tau) = 1 + 1/f(0) \quad . \tag{S2}$$

We set $f(0) = 2N/3 >> 1$, $f(\varepsilon) = N/3 >> 1$ and divide (S1) by (S2),

$$\exp(\varepsilon/\tau) = (1+3/N)/(1+3/2N) \cong 1+3/2N \quad ,$$

where in the last equality we utilized $3/2N << 1$. Taking logarithms, and again using $3/2N << 1$:

$$\varepsilon/\tau = \log(1+3/2N) \cong 3/2N; \quad \tau \cong 2N\varepsilon/3 \quad .$$

Note that $N\varepsilon$ occurs where we might naively have expected a quantity of order ε. This example illustrates once again a point made in the text, that the Bose-Einstein condensation of the majority of particles into the lowest orbital takes place at a temperature much larger than the energy separation between the lowest and next-lowest orbitals.