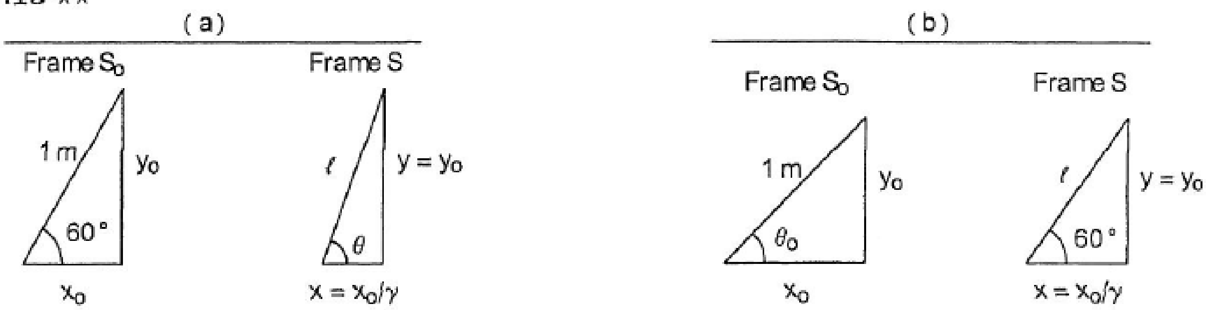
Homework #9, Problem: 15.5, 15.13, 15.19, 15.21, 15.24, 15.38, 15.44

15.5 ⋆ With $\beta = 0.95$, the γ factor for both the outward and return trips is $\gamma = 1/\sqrt{1-\beta^2} = 3.20$. The times for the two halves of the journey satisfy

$$\Delta t_B^{\text{out}} = \gamma \Delta t_A^{\text{out}} \quad \text{and} \quad \Delta t_B^{\text{back}} = \gamma \Delta t_A^{\text{back}},$$

so, by addition, the times for the whole journey satisfy the same relation $\Delta t_B = \gamma \Delta t_A$. Therefore $\Delta t_A = \Delta t_B/\gamma = (80 \text{ yr})/3.20 = 25$ yr, which is the amount by which twin A has aged.

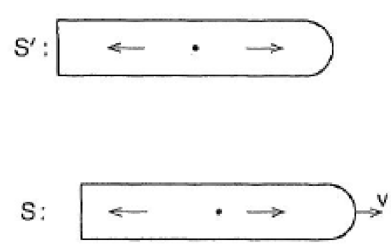
15.13 ⋆⋆



(a) With $\beta = 4/5$, $\gamma = 5/3$. In the frame $\mathcal{S}_o$, we know the length $l_o = 100$ cm and the angle $\theta_o = 60°$, so we can calculate $x_o = 50$ cm and $y_o = 86.6$ cm. In the frame $\mathcal{S}$, x is contracted ($x = x_o/\gamma = 30$ cm) but y is not ($y = y_o = 86.6$ cm). Thence $l = \sqrt{x^2+y^2} = 91.7$ cm and $\theta = \arctan(y/x) = 70.9°$.

(b) The angle 60° is given in the frame $\mathcal{S}$, so $\tan 60° = y/x = y_o/(x_o/\gamma)$ and $\tan\theta_o = y_o/x_o = (\tan 60°)/\gamma$, whence $\theta_o = 46.1°$. From this we find $x_o = 69.3$ cm and $y_o = 72.1$ cm, and from these we can calculate $x = x_o/\gamma$ and $y = y_o$ and thence $l = 83.2$ cm.

15.19 ⋆⋆ (a) $x'_F = d$, $t'_F = d/c$; $x'_B = -d$, $t'_B = d/c$.



(b)

$$x_F = \gamma(x'_F + vt'_F) = \gamma(1+\beta)d$$
$$t_F = \gamma(t'_F + vx'_F/c^2) = \gamma(1+\beta)d/c$$
$$x_B = \gamma(x'_B + vt'_B) = -\gamma(1-\beta)d$$
$$t_B = \gamma(t'_B + vx'_B/c^2) = \gamma(1-\beta)d/c$$

Although the two events are simultaneous as measured in S', they are *not* simultaneous in S. As observed in S, the two signals start out from the middle of the rocket, but while they are traveling the rocket is also traveling to the right at speed v. Thus the front is receding from its signal, which must travel more than half the rocket's length. Meanwhile the back of the rocket is approaching its signal, which needs to travel only a shorter distance. Therefore this signal arrives first; that is, $t_B < t_F$. (In S' the signals again start from the middle of the rocket; but since the rocket is not moving they naturally arrive simultaneously.)

15.21 ⋆ Let us take our x axis in the direction of the two velocities. Then the velocity of the rocket's frame $\mathcal{S}'$ has $V = \frac{1}{2}c$ and that of the bullets relative to the rocket has $v'_x = \frac{3}{4}c$, with all other components zero. According to the inverse of the velocity-addition formula (15.26),

$$v_x = \frac{v'_x + V}{1 + v'_x V/c^2} = \frac{\frac{1}{2} + \frac{3}{4}}{1 + \frac{3}{8}}\,c = \frac{5/4}{11/8}\,c = \frac{10}{11}\,c,$$

with all other components zero.

15.24 ⋆ Let $\mathcal{S}$ be the frame fixed to the ground and $\mathcal{S}'$ the one fixed to the cop's car. The velocity of $\mathcal{S}'$ relative to $\mathcal{S}$ is $V = 0.4c$ and the velocity of the bullet relative to $\mathcal{S}'$ is $v = 0.5c$. By the inverse velocity transformation, the bullet's velocity relative to the ground is

$$v = \frac{v' + V}{1 + v'V/c^2} = \frac{0.4 + 0.5}{1 + 0.4 \times 0.5}\,c = 0.75c.$$

Since the robber's velocity relative to the ground is $0.8c$, the bullets do not catch the robber.

15.38 ⋆⋆ In the observer's rest frame, $dx = (0, 0, 0, c\,dt)$. On the other hand, since P and Q are simultaneous, $t_P = t_Q$ and $x_P - x_Q = (\Delta\mathbf{x}, 0)$. Therefore, $(x_P - x_Q)\cdot dx = 0$.

15.44 ⋆⋆ (a) If q is time-like, then $q{\cdot}q = |\mathbf{q}|^2 - q_4^2 < 0$, which implies that $|\mathbf{q}| < |q_4|$. First rotate the coordinates so that $\mathbf{q}$ points along the x axis and $q = (q_1, 0, 0, q_4)$, with $|q_1| < |q_4|$. Now apply the standard boost to give $q'_1 = \gamma(q_1 - \beta q_4)$. We can choose $\beta = q_1/q_4$ (since $|q_1| < |q_4|$, this makes $|\beta| < 1$, as it has to be) and then $q'_1 = 0$ and $q' = (0, 0, 0, q'_4)$.

(b) A vector q is forward time-like if and only if $q^2 < 0$ and $q_4 > 0$. The first condition is Lorentz invariant and implies that $|\mathbf{q}| < q_4$. Now suppose, in addition, the second condition holds in a frame $\mathcal{S}$ and imagine applying a standard boost so that, in the new frame $\mathcal{S}'$, $q'_4 = \gamma(q_4 - \beta q_1)$. Now, $|\beta| < 1$ and $|q_1| \le |\mathbf{q}| < q_4$. Therefore, $q'_4 > 0$ and the second condition is valid in $\mathcal{S}'$ also. This conclusion would certainly not be changed if we made any rotation of our coordinates, and since any Lorentz transformation can be built up from standard boosts and rotations, $q'_4 > 0$ in any inertial frame. Therefore a vector that is forward time-like in one frame is forward time-like in all frames.