Homework #8, Problems: 14.11, 14.14, 14.18, 14.24, 14.27, 14.28, 14.29

14.11 ★★ The target density is

$$n_{\text{tar}} = \frac{\varrho t}{m_{\text{Ag}}} = \frac{(10.5 \times 10^3 \text{ kg/m}^3) \times (10^{-6} \text{ m})}{108 \times (1.66 \times 10^{-27} \text{ kg})} = 5.86 \times 10^{22} \text{ m}^{-2}$$

and the solid angle subtended by the counter is $\Delta\Omega = (0.1 \text{ mm}^2)/(10 \text{ mm})^2 = 10^{-3}$ sr. Therefore, the number of alphas scattered into $\Delta\Omega$ should be

$$N_{\text{sc}} = N_{\text{inc}} n_{\text{tar}} \frac{d\sigma}{d\Omega}\Delta\Omega = (10^{10}) \times (5.86 \times 10^{22} \text{ m}^{-2}) \times (0.5 \times 10^{-28} \text{ m}^2/\text{sr}) \times (10^{-3} \text{ sr}) \approx 29.$$

14.14 ★★ **(a)** If we consider Fig.14.1 to show a two-dimensional scattering event, then on the one hand the definition of the differential cross section (really differential width) is that the number of scatterings between θ and $\theta + d\theta$ is

$$N_{\text{sc}} = N_{\text{inc}} n_{\text{tar}} \frac{d\sigma}{d\theta} d\theta,$$

where n_{tar} is the density (number/width) of targets. [This is the analog of Eq.(14.17) for two-dimensional scattering.] On the other hand, by the familiar argument (modified for two dimensions) $N_{\text{sc}} = N_{\text{inc}} n_{\text{tar}} db$. Comparing these two equations, we see that $d\sigma/d\theta = |db/d\theta|$, where the absolute values signs are because $d\theta$ and db were both taken to be positive.

(b) From Fig.14.10 (considered as a two-dimensional picture) we see that $b = R\sin\alpha$ and that $\theta = \pi - 2\alpha$ or $\alpha = \frac{1}{2}(\pi - \theta)$. Therefore $b = R\sin\frac{1}{2}(\pi - \theta) = R\cos(\theta/2)$, and

$$\frac{d\sigma}{d\theta} = \left|\frac{db}{d\theta}\right| = \frac{R}{2}\left|\sin(\theta/2)\right|.$$

(c) $\sigma = \int_{-\pi}^{\pi} \frac{d\sigma}{d\theta} d\theta = 2\int_0^{\pi} (R/2)\sin(\theta/2) d\theta = 2R \int_0^{\pi/2} \sin(u) du = 2R.$

14.18 ⋆⋆ The cross section for backward scattering is found by integrating $d\sigma/d\Omega = \sigma_o/\sin^4(\theta/2)$ over the backward hemisphere, $\theta \geq \pi/2$:

$$\sigma(\theta \geq \pi/2) = \int_{\theta\geq\pi/2} d\Omega \frac{d\sigma}{d\Omega} = \sigma_o \int_0^{2\pi} d\phi \int_{\pi/2}^{\pi} \frac{\sin\theta\, d\theta}{\sin^4(\theta/2)} = 4\pi\sigma_o.$$

[One way to do the integral over θ is to write $\sin\theta$ as $2\sin(\theta/2)\cos(\theta/2)$ and make the substitution $u = \sin(\theta/2)$.]

The number of α's scattered into the backward hemisphere should be $N_{sc}(\theta \geq \pi/2) = N_{inc} n_{tar} \sigma(\theta \geq \pi/2)$. Thus the requested ratio is $N_{sc}(\theta \geq \pi/2)/N_{inc} = n_{tar}\sigma(\theta \geq \pi/2)$. With

$$n_{tar} = \frac{\varrho t}{m_{Pt}} = \frac{(21.4 \times 10^3 \text{ kg/m}^3) \times (3 \times 10^{-6} \text{ m})}{195 \times 1.66 \times 10^{-27} \text{ kg}} = 1.98 \times 10^{23} \text{ m}^{-2}$$

and

$$\sigma_o = \left(\frac{kqQ}{4E}\right)^2 = \left(\frac{(8.99 \times 10^9 \text{ N}\cdot\text{m}^2/\text{ C}^2) \times 2 \times 78 \times (1.60 \times 10^{-19} \text{ C})^2}{4 \times (7.8 \times 1.60 \times 10^{-13} \text{ J})}\right)^2 = 5.17\times10^{-29} \text{ m}^2$$

the required ratio is $n_{tar}\sigma(\theta \geq \pi/2) = 4\pi n_{tar}\sigma_o = 1.29 \times 10^{-4}$. The reciprocal of this is 7750, so the Rutherford model predicts that 1 particle in 7750 would be "reflected" into the backward hemisphere, in remarkable agreement with Geiger and Marsden's observed "about 1 in 8000."

14.24 ⋆⋆ **(a)** We are told that, if $m_1 = m_2$, then $\theta_{lab} = \frac{1}{2}\theta_{cm}$. To relate the corresponding differential cross sections we need to find the derivative

$$\frac{d(\cos\theta_{cm})}{d(\cos\theta_{lab})} = \frac{d(\cos 2\theta_{lab})}{d(\cos\theta_{lab})} = \frac{d(2\cos^2\theta_{lab} - 1)}{d(\cos\theta_{lab})} = 4\cos\theta_{lab}.$$

Substitution into (14.45) yields the claimed result (14.63).

(b) Since $(d\sigma/d\Omega)_{cm} = R^2/4$, it follows that $(d\sigma/d\Omega)_{lab} = (4\cos\theta_{lab})R^2/4 = R^2\cos\theta_{lab}$. This is for $0 \leq \theta_{lab} \leq \pi/2$. Since no particles are scattered with $\theta_{lab} > \pi/2$, it follows that $(d\sigma/\Omega)_{lab} = 0$ for $\pi/2 < \theta_{lab} \leq \pi$. Therefore,

$$\sigma_{tot} = \int d\Omega_{lab} \left(\frac{d\sigma}{d\Omega}\right)_{lab} = R^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos(\theta_{lab})\sin(\theta_{lab}) d\theta_{lab} = R^2 \cdot 2\pi \cdot \int_0^1 u\, du = \pi R^2,$$

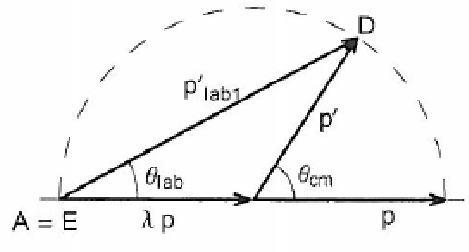
where the last integral results from the substitution $u = \sin\theta_{lab}$.

14.27 ⋆⋆ **(a)** With $\lambda = 1$, Eq.(14.64) reads

$$\tan\theta_{\text{lab}} = \frac{\sin\theta_{\text{cm}}}{1+\cos\theta_{\text{cm}}} = \frac{2\sin(\theta_{\text{cm}}/2)\cos(\theta_{\text{cm}}/2)}{1+\cos^2(\theta_{\text{cm}}/2)-\sin^2(\theta_{\text{cm}}/2)} = \frac{\sin(\theta_{\text{cm}}/2)}{\cos(\theta_{\text{cm}}/2)} = \tan(\theta_{\text{cm}}/2).$$

Therefore, $\theta_{\text{lab}} = \frac{1}{2}\theta_{\text{cm}}$.

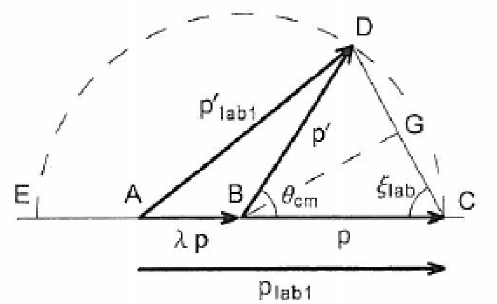
(b) Compare the figure here with Figure 14.15. Here, with $\lambda = 1$, the point A has moved out and coincides with E. If $m_1 < m_2$ (Fig. 14.15), then as θ_{cm} increases toward π the point D moves around toward E and $\theta_{\text{lab}} \to \pi$. But here, with equal masses, as θ_{cm} increases toward π the line AD becomes tangent to the circle at A and $\theta_{\text{lab}} \to \pi/2$.



14.28 ⋆⋆ **(a)** In the lab frame, the initial momentum of particle 2 is zero, $\mathbf{p}_{\text{lab2}} = 0$. Therefore, conservation of momentum implies that

$$\mathbf{p}_{\text{lab1}} = \mathbf{p}'_{\text{lab1}} + \mathbf{p}'_{\text{lab2}}. \tag{iii}$$

That is, these three vectors form a triangle in the appropriate order. Since the first two vectors are represented by the sides AC and AD, it follows that $\mathbf{p}'_{\text{lab2}}$ is represented by the side DC. The angle ξ_{lab} is the angle between $\mathbf{p}'_{\text{lab2}}$ and the incident direction, and this is equal to the angle BCD. By looking at the right triangle BCG, you should be able to convince yourself that $\xi_{\text{lab}} = (\pi/2) - (\theta_{\text{cm}}/2)$, as claimed.



(b) If the masses are equal, then $\lambda = 1$ and the point A in Fig.14.15 coincides with E; that is, AC is a diameter, and, by a well known theorem of geometry, the angle ADC is 90°.

(c) If you square Eq.(iii) above (conservation of momentum) and compare the result with the equation of conservation of kinetic energy, $T_1 = T_1' + T_2'$, you will find that $\mathbf{p}'_{\text{lab1}} \cdot \mathbf{p}'_{\text{lab2}} = 0$, which says that the two final momenta are perpendicular.

14.29 ⋆⋆ **(a)** In the CM frame the total momentum is zero, so the two initial momenta are equal in magnitude, $p_1 = p_2$ and likewise the two final momenta, $p'_1 = p'_2$. This let's us write the conservation of kinetic energy (elastic collision) as

$$E = p_1^2\left(\frac{1}{2m_1}+\frac{1}{2m_2}\right) = E' = p_1'^2\left(\frac{1}{2m_1}+\frac{1}{2m_2}\right)$$

which implies that $p_1 = p'_1$ and hence that the KE of particle 1 is separately conserved, and likewise for particle 2.

(b) In the lab frame, particle 2 is initially at rest, so that $T_2 = 0$. If any kind of collision occurs, particle 2 must recoil with $T_2 > 0$. Therefore, particle 2 *gains* kinetic energy. By conservation of energy, the projectile must lose KE, and the separate energies are definitely *not* conserved.

(c) In Fig.14.15 you can see that the final momentum of particle 2 is represented by the line DC, which has magnitude $p'_{\rm lab2} = 2p\sin(\theta_{\rm cm}/2)$. Therefore the energy gained by particle 2 (and lost by particle 1) is

$$\Delta E = \frac{(p'_{\rm lab2})^2}{2m_2} = \frac{2p^2\sin^2(\theta_{\rm cm}/2)}{m_2}.$$

This is to be compared with the original energy of particle 1, which is (in the lab frame)

$$E = T_1 = \frac{(p_{\rm lab1})^2}{2m_1} = \frac{(1+\lambda)^2p^2}{2m_1}$$

Therefore

$$\frac{\Delta E}{E} = \frac{2p^2\sin^2(\theta_{\rm cm}/2)}{m_2}\cdot\frac{2m_1}{(1+\lambda)^2p^2} = \frac{4\lambda\sin^2(\theta_{\rm cm}/2)}{(1+\lambda)^2}.$$

(d) For given λ this fractional loss is greatest if $\theta_{\rm cm}/2 = 90°$ or $\theta_{\rm cm} = 180°$ — what one would normally expect in a direct head-on collision. Differentiating with respect to λ, you can easily check that the corresponding fractional loss is maximum if $\lambda = 1$, that is, if the two particles have equal masses.