Homework #7, Problems: 14, 19, 26, 28, 32 in chapter 11

11.14 ★★ **(a)** The kinetic energy is $T = \frac{1}{2}mL^2(\dot\phi_1^2 + \dot\phi_2^2)$. The gravitational potential energy of either pendulum has the form $mgL(1 - \cos\phi) \approx \frac{1}{2}mgL\phi^2$, and the spring's PE is $\frac{1}{2}kx^2 \approx \frac{1}{2}kL^2(\phi_2 - \phi_1)^2$. Putting these together,

$$\mathcal{L} = \tfrac{1}{2}mL^2(\dot\phi_1^2 + \dot\phi_2^2) - \tfrac{1}{2}mgL(\phi_1^2 + \phi_2^2) - \tfrac{1}{2}kL^2(\phi_2 - \phi_1)^2$$

from which we get the Lagrange equations:

$$\begin{aligned} \ddot\phi_1 &= -\omega_o{}^2\phi_1 + (k/m)(\phi_2 - \phi_1) \\ \ddot\phi_2 &= -\omega_o{}^2\phi_2 - (k/m)(\phi_2 - \phi_1) \end{aligned}$$

where I have divided through by mL^2 and introduced the natural frequency for either pendulum (without the spring) given by $\omega_o{}^2 = g/L$.

(b) From the equations of motion, you can write down the "mass matrix" $\mathbf{M}$ and "spring matrix" $\mathbf{K}$, and thence the matrix

$$\mathbf{K} - \omega^2\mathbf{M} = \begin{bmatrix} \omega_o{}^2 + k/m - \omega^2 & -k/m \\ -k/m & \omega_o{}^2 + k/m - \omega^2 \end{bmatrix}.$$

The determinant of this matrix is $(\omega_o{}^2 - \omega^2)(\omega_o{}^2 + 2k/m - \omega^2)$, and the two normal frequencies are

$$\omega_1 = \omega_o \qquad \text{and} \qquad \omega_2 = \sqrt{\omega_o{}^2 + 2k/m}.$$

The corresponding motions are found by solving the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ with ω set equal to ω_1 and ω_2 in turn. For the first mode, this gives the eigenvector $\mathbf{a} = A(1, 1)$ (actually a 2×1 column, of course). This means that in the first mode the two pendulums oscillate in unison (in phase with equal amplitudes). In this mode the spring is unstretched, its presence is irrelevant, and the frequency is just the natural frequency for a single pendulum.

For the second mode, $\mathbf{a} = A(1, -1)$, and the two pendulums oscillate with equal amplitudes but exactly out of phase. Notice that, in either mode, the two pendulums behave just like the two carts of Section 11.2.

11.19 ⋆⋆⋆ **(a)** The PE (measured from the equilibrium positions) is

$$U = \tfrac{1}{2}kx^2 + MgL(1 - \cos\phi) \approx \tfrac{1}{2}kx^2 + \tfrac{1}{2}MgL\phi^2.$$

The KE is a bit trickier because the velocity of the bob M is the vector sum of the velocity $\dot{x}$ of the support and $L\dot{\phi}$ of the bob relative to the support. In general these are in different directions, but as long as ϕ remains small they are essentially parallel, and the KE is

$$T \approx \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}M(\dot{x} + L\dot{\phi})^2 = \tfrac{1}{2}(m + M)\dot{x}^2 + ML\dot{x}\dot{\phi} + \tfrac{1}{2}ML^2\dot{\phi}^2.$$

From these you can write down the Lagrangian $\mathcal{L}$ and the two Lagrange equations. The x equation is

$$(m + M)\ddot{x} + ML\ddot{\phi} = -kx$$

and the ϕ equation is

$$ML\ddot{x} + ML^2\ddot{\phi} = -MgL\phi.$$

We can write these as a single matrix equation, $\mathbf{M}\ddot{\mathbf{q}} = -\mathbf{K}\mathbf{q}$, if we define

$$\mathbf{q} = \begin{bmatrix} x \\ \phi \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m + M & ML \\ ML & ML^2 \end{bmatrix}, \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} k & 0 \\ 0 & MgL \end{bmatrix}.$$

(b) With the given values, the matrix $(\mathbf{K} - \omega^2\mathbf{M})$ is

$$\mathbf{K} - \omega^2\mathbf{M} = \begin{bmatrix} 2 - 2\omega^2 & -\omega^2 \\ -\omega^2 & 1 - \omega^2 \end{bmatrix}$$

which has determinant $\det(\mathbf{K} - \omega^2\mathbf{M}) = 2(1 - \omega^2)^2 - \omega^4 = \omega^4 - 4\omega^2 + 2$. The normal frequencies are the zeros of this determinant and are

$$\omega_1 = \sqrt{2 - \sqrt{2}} = 0.77 \quad \text{and} \quad \omega_2 = \sqrt{2 + \sqrt{2}} = 1.85.$$

The motion in each corresponding mode is given by the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$. For the first mode, this gives $a_2 = \sqrt{2}a_1$, so the cart and bob oscillate in phase (both moving to the right and then both moving to the left), with the bob's amplitude (of motion relative to the cart) $\sqrt{2}$ times bigger than the cart's. For the second mode, $a_2 = -\sqrt{2}a_1$, so the cart and bob oscillate exactly out of phase, again with the amplitude of the bob equal to $\sqrt{2}$ times that of the cart.

11.26 ⋆⋆ The moment of inertia of the hoop about its edge is $I = 2mR^2$, so its KE is $T_1 = \frac{1}{2}I\dot{\phi}_1^2 = mR^2\dot{\phi}_1^2$. The speed of the bead is (for small oscillations) just $v_2 = R(\dot{\phi}_1 + \dot{\phi}_2)$, so its KE is $T_2 = \frac{1}{2}mR^2(\dot{\phi}_1 + \dot{\phi}_2)^2$. Therefore

$$T = \tfrac{1}{2}mR^2(3\dot{\phi}_1^2 + 2\dot{\phi}_1\dot{\phi}_2 + \dot{\phi}_2^2).$$

The total PE is

$$U = U_1 + U_2 = mgR(1 - \cos\phi_1) + mgR[(1 - \cos\phi_1) + (1 - \cos\phi_2)] \approx \tfrac{1}{2}mgR(2\phi_1^2 + \phi_2^2)$$

for small oscillations. Therefore the matrices **M** and **K** are

$$\mathbf{M} = mR^2\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{K} = mR^2\omega_o^2\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

where $\omega_o = \sqrt{g/R}$, the frequency of a pendulum of length R. From these you can check that $\det(\mathbf{K} - \omega^2\mathbf{M}) = m^2R^4(2\omega^2 - \omega_o^2)(\omega^2 - 2\omega_o^2)$, so that the natural frequencies are given by $\omega_1^2 = \frac{1}{2}\omega_o^2$ and $\omega_2^2 = 2\omega_o^2$. If we substitute $\omega = \omega_1$ into the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$, we find $a_1 = a_2$. Thus in the first mode the two angles oscillate in phase with equal amplitudes; that is, the bead and hoop oscillate together, with the bead stationary relative to the hoop. Substituting $\omega = \omega_2$, we find $a_2 = -2a_1$ so the bead and hoop oscillate 180° out of phase with the amplitude of ϕ_2 twice that of ϕ_1.

11.28 ⋆⋆ **(a)** For small oscillations, the KE is

$$T = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}M(\dot{x} + L\dot{\phi}^2) = \tfrac{1}{2}(m + M)\dot{x}^2 + ML\dot{x}\dot{\phi} + \tfrac{1}{2}ML^2\dot{\phi}^2$$

while the PE is $U = MgL(1 - \cos\phi) \approx \frac{1}{2}MgL\phi^2$. Thus if we take x and ϕ as our coordinates (in that order) the matrices **M** and **K** are

$$\mathbf{M} = \begin{bmatrix} m + M & ML \\ ML & ML^2 \end{bmatrix} = M\begin{bmatrix} 1 + \lambda & L \\ L & L^2 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} 0 & 0 \\ 0 & MgL \end{bmatrix} = M\begin{bmatrix} 0 & 0 \\ 0 & L^2\omega_o^2 \end{bmatrix}$$

where I have introduced the mass ratio $\lambda = m/M$ and the frequency $\omega_o = \sqrt{g/L}$. Therefore

$$(\mathbf{K} - \omega^2\mathbf{M}) = -M\begin{bmatrix} \omega^2(1 + \lambda) & \omega^2 L \\ \omega^2 L & (\omega^2 - \omega_o^2)L^2 \end{bmatrix}$$

and (as you can check) $\det(\mathbf{K} - \omega^2\mathbf{M}) = ML^2\omega^2[\lambda\omega^2 - (1 + \lambda)\omega_o^2]$. The normal frequencies are $\omega_1 = 0$ and $\omega_2 = \omega_o\sqrt{(1 + \lambda)/\lambda}$.

(b) If we set $\omega = \omega_1 = 0$, the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ reduces to $\mathbf{Ka} = 0$, which requires that $a_2 = 0$; that is, $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (times any constant). If we try a solution of the form $\mathbf{x}(t) = \mathbf{a}f(t)$, then the equation of motion $\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{Kx}$ becomes $\ddot{f} = 0$, so that $f(t) = x_o + v_o t$. In this mode, the cart moves with constant velocity, while the pendulum is stationary relative to the cart.

If we set $\omega = \omega_2$, the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ requires that $(1 + \lambda)a_1 = -La_2$, so $x = A\cos(\omega_2 t - \delta)$ and $\phi = -A\cos(\omega_2 t - \delta)(1 + \lambda)/L$. In this mode, the cart and bob oscillate in opposite directions leaving their CM stationary.

11.32 ⋆⋆⋆ I'll introduce the notation $\lambda = M/m$ for the ratio of the two different masses, and I'll use units with $m = k = 1$. With this arrangement, any frequencies are measured in units of $\omega_o = \sqrt{k/m}$, the natural frequency of a mass m on a spring k. The total KE is $T = \frac{1}{2}(\dot{x}_1^2 + \lambda\dot{x}_2^2 + \dot{x}_3^2)$ and the PE is $U = \frac{1}{2}k(x_1 - x_2)^2 + \frac{1}{2}k(x_2 - x_3)^2$. The matrices $\mathbf{M}$ and $\mathbf{K}$ are

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

and

$$\mathbf{K} - \omega^2\mathbf{M} = \begin{bmatrix} 1-\omega^2 & -1 & 0 \\ -1 & 2-\lambda\omega^2 & -1 \\ 0 & -1 & 1-\omega^2 \end{bmatrix}.$$

Therefore, as you can check, $\det(\mathbf{K} - \omega^2\mathbf{M}) = -\omega^2(\omega^2 - 1)(\lambda\omega^2 - 2 - \lambda)$ and the normal frequencies are $\omega_1 = 0$, $\omega_2 = 1$, and $\omega_3 = \sqrt{(2+\lambda)/\lambda}$. (In arbitrary units the last two would be $\omega_2 = \omega_o$, and $\omega_3 = \sqrt{(2+\lambda)/\lambda}\,\omega_o$.)

(b) If we put $\omega = \omega_2 = 1$ in the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$, we find that $a_2 = 0$ and $a_1 = -a_3$. Thus, in the second mode the center atom is stationary, while the outer two oscillate with frequency ω_o and equal amplitudes but 180° out of phase. If we put $\omega = \omega_3$ in the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$, we find that $a_1 = a_3 = -a_2(\lambda/2)$. Thus, in the third mode, atoms 1 and 3 oscillate in phase with the same amplitude, while the center atom oscillates 180° out of phase with amplitude $2/\lambda$ times that of the others.

(c) If we put $\omega = 0$, the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ becomes $\mathbf{Ka} = 0$, and we find that $a_1 = a_2 = a_3$. As in Problem 11.27, if we try a solution of the form $\mathbf{x} = \mathbf{a}f(t)$, the eqution of motion implies that $\ddot{f} = 0$, so all three atoms move with the same constant velocity separated by their equilibrium separation.