Homework #5, problem 11.2, 11.4, 11.5, 11.12

11.2 ⋆⋆ Let x_1 and x_2 be the extensions of the two springs from their unstretched lengths, and x_{10} and x_{20} their values at equilibrium. The displacements from equilibrium are

$$y_1 = x_1 - x_{10} \quad \text{and} \quad y_2 = x_2 - x_{20}. \tag{i}$$

The net downward forces on the two masses are

$$F_1 = m_1 g - k_1 x_1 + k_2(x_2 - x_1) \quad \text{and} \quad F_2 = m_2 g - k_2(x_2 - x_1), \tag{ii}$$

and the conditions for equilibrium are

$$m_1 g = k_1 x_{10} - k_2(x_{20} - x_{10}) \quad \text{and} \quad m_2 g = k_2(x_{20} - x_{10}).$$

Using (i) to eliminate x_1 and x_2 from (ii), we find that

$$\begin{aligned} m_1 \ddot{y}_1 = F_1 &= m_1 g - k_1(y_1 + x_{10}) + k_2[y_2 - y_1 + (x_{20} - x_{10})] \\ &= -k_1 y_1 + k_2(y_2 - y_1) \end{aligned}$$

where, in the second line I used the equilibrium consition to cancel several terms. Similarly

$$\begin{aligned} m_2 \ddot{y}_2 = F_2 &= m_2 g - k_2(x_2 - x_1) \\ &= -k_2(y_2 - y_1). \end{aligned}$$

The last two results combine to give the matrix equation $\mathbf{M}\ddot{\mathbf{y}} = -\mathbf{K}\mathbf{y}$ where

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

11.4 ⋆⋆ **(a)** Putting $m_1 = m_2$ and $k_1 = k_3$ in (11.5), we find that the mass and spring-constant matrices are

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \qquad \text{and} \qquad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix}$$

and hence

$$(\mathbf{K} - \omega^2\mathbf{M}) = \begin{bmatrix} k_1 + k_2 - m\omega^2 & -k_2 \\ -k_2 & k_1 + k_2 - m\omega^2 \end{bmatrix}.$$

The determinant of the last matrix is $\det(\mathbf{K} - \omega^2\mathbf{M}) = (m\omega^2 - k_1)(m\omega^2 - k_1 - 2k_2)$. The two normal frequencies are the roots of the equation $\det(\mathbf{K} - \omega^2\mathbf{M}) = 0$ and are $\sqrt{k_1/m}$ and $\sqrt{(k_1 + 2k_2)/m}$. If I set $k_1 = k_2 = k$, these reduce to the results (10.15) for all three springs equal.

(b) The motion in each normal mode is determined by the vector $\mathbf{a}$ satisfying the eigenvector equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$. For $\omega = \omega_1$ this is easily seen to be exactly the same as for the equal-spring case; in particular, the motion is as given by (11.18) and as shown in Figure 11.2. The two carts oscillate in phase with equal amplitudes, so that the middle spring is undisturbed. This means its strength is irrelevant and we get the same motion with the same frequency whatever the value of k_2. For the second mode, with $\omega = \omega_2$, the motion is again the same as for the corresponding mode of the equal-spring case, namely (11.20) and Figure 11.4. This is a little subtle: In this mode, the middle spring does change length, and the frequency does depend on the value of k_2. Nevertheless, the motion is independent of k_2 since the symmetric arrangement, with the outside springs equally stretched and the middle one compressed, or vice versa, leads to equal (but opposite) forces on the two equal-mass carts and allows them to oscillate with equal amplitudes exactly out of phase.

11.5 ⋆⋆ **(a)** The quickest way to find the equation of motion for the system of Fig.11.15 is to set $k_3 = 0$ in Fig.11.1. With $m_1 = m_2$ and $k_1 = k_2$ as well, the mass and spring-constant matrices are given by Eq.(11.5) as

$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \qquad \text{and} \qquad \mathbf{K} = \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}.$$

If we define $\omega_o = \sqrt{k/m}$ and hence $k = m\omega_o^2$, the characteristic equation becomes

$$\det(\mathbf{K} - \omega^2\mathbf{M}) = m^2(\omega^4 - 3\omega_o^2\omega^2 + \omega_o^4) = 0,$$

so the normal frequencies are given by $\omega^2 = \omega_o^2(3 \pm \sqrt{5})/2$.

(b) If we substitute $\omega = \omega_1 = \omega_o\sqrt{(3 - \sqrt{5})/2}$, the equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ yields $a_2 = a_1(1 + \sqrt{5})/2 = 1.62a_1$. Thus the first normal mode has the form

$$x_1 = A\cos(\omega_1 t - \delta) \qquad \text{and} \qquad x_2 = 1.62A\cos(\omega_1 t - \delta).$$

where A and δ are arbitrary constants. In the first mode the two carts oscillate in phase, the second one with the larger amplitude. Similarly, in the second mode, we find $a_2 = a_1(1-\sqrt{5})/2 = -0.62a_1$ and so

$$x_1 = A\cos(\omega_2 t - \delta) \qquad \text{and} \qquad x_2 = -0.62A\cos(\omega_2 t - \delta).$$

where (in general) A and δ are different constants. In this mode the two carts move 180° out of phase, and cart 2 has the smaller amplitude.

11.12 ★★★ **(a)** The force of viscous drag is $\beta m(\dot{x}_2 - \dot{x}_1)$, to the right on cart 1 and to the left on cart 2. The equation of motion for cart 1 is

$$m\ddot{x}_1 = -kx_1 + \beta m(\dot{x}_2 - \dot{x}_1) \quad \text{or} \quad \ddot{x}_1 + \omega_o^2 x_1 + \beta\dot{x}_1 - \beta\dot{x}_2 = 0$$

and that for cart 2 is

$$m\ddot{x}_2 = -kx_2 - \beta m(\dot{x}_2 - \dot{x}_1) \quad \text{or} \quad \ddot{x}_2 + \omega_o^2 x_2 - \beta\dot{x}_1 + \beta\dot{x}_2 = 0.$$

These two equations combine as a single matrix equation $\ddot{\mathbf{x}} + \omega_o^2\mathbf{x} + \beta\mathbf{D}\dot{\mathbf{x}} = 0$, where $\mathbf{D}$ and $\mathbf{x}$ are the matrices

$$\mathbf{D} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(b) If we substitute the proposed complex solution $\mathbf{z}(t) = \mathbf{a}e^{rt}$ into the matrix equation of motion, we find the $\mathbf{a}$ must satisfy

$$[(r^2 + \omega_o^2)\mathbf{1} + \beta r\mathbf{D}]\mathbf{a} = 0. \tag{viii}$$

This has nontrivial solutions only if the deteminant of the matrix in brackets is zero. That is,

$$\det[(r^2 + \omega_o^2)\mathbf{1} + \beta r\mathbf{D}] = (r^2 + \omega_o^2)(r^2 + \omega_o^2 + 2\beta r) = 0$$

Thus the values of r that give a solution are $r = r_1 = i\omega_o$ and $r = r_2 = -\beta + i\sqrt{\omega_o^2 - \beta^2} = -\beta + i\omega_1$. [There are actually two more solutions with the opposite sign to the imaginary part, but these give the same actual motion $\mathbf{x}(t)$.] If we put $r = r_1$ in (viii) we find that $a_1 = a_2 = A$, say. Thus the first mode has $x_1(t) = x_2(t) = A\cos(\omega_o t - \delta)$, and the two carts move together with equal amplitudes. Because cart 2 is stationary with respect to cart 1, the drag force is zero and the motion is undamped. If we put $r = r_2$ in (viii) we find that $a_1 = -a_2 = A$, say, and the second mode has $x_1(t) = -x_2(t) = A\cos(\omega_1 t - \delta)e^{-\beta t}$. In this mode the two carts move in opposite directions and the drag force causes the motion to damp out.