Homework #4, Problem 10.40, 10.42, 10.44, 10.45

10.40 ⋆⋆ **(a)** Multipying the first of Equations (10.86) by $\lambda_1\omega_1$, the left side becomes $\lambda_1^2\omega_1\dot{\omega}_1$, which is the same as $\frac{1}{2}d(\lambda_1^2\omega_1^2)/dt$. Therefore

$$\frac{d}{dt}\left(\lambda_1^2\omega_1^2\right) = 2\lambda_1(\lambda_2 - \lambda_3)\omega_1\omega_2\omega_3.$$

Similarly, the second and third equations give

$$\frac{d}{dt}\left(\lambda_2^2\omega_2^2\right) = 2\lambda_2(\lambda_3 - \lambda_1)\omega_1\omega_2\omega_3 \quad \text{and} \quad \frac{d}{dt}\left(\lambda_3^2\omega_3^2\right) = 2\lambda_3(\lambda_1 - \lambda_2)\omega_1\omega_2\omega_3.$$

Adding these three equations and remembering that $\mathbf{L} = (\lambda_1\omega_1, \lambda_2\omega_2, \lambda_3\omega_3)$, we find that $d\mathbf{L}^2/dt = 0$.

(b) If, instead, we multiply the first of Equations (10.88) by ω_1, we find that

$$\frac{1}{2}\frac{d}{dt}\left(\lambda_1\omega_1^2\right) = (\lambda_2 - \lambda_3)\omega_1\omega_2\omega_3.$$

Adding this to the corresponding two equations for the second and third components, we find that

$$\frac{1}{2}\frac{d}{dt}\left(\lambda_1\omega_1^2 + \lambda_2\omega_2^2 + \lambda_3\omega_3^2\right) = \frac{d}{dt}T_{\text{rot}} = 0.$$

10.42 ⋆ The inertia tensor for the book (sides $a = 30$, $b = 20$, and $c = 3$, all in cm) can be evaluated as in Example 10.2. With the origin at the CM, all off-diagonal elements are zero, and the diagonal elements (which are the principal moments) are $\lambda_1 = M(b^2 + c^2)/12$, and so on. If the book's spin axis is close to the shortest symmetry axis (the z axis), then according to (10.91) the frequency of wobble is given by

$$\Omega^2 = \frac{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}{\lambda_1\lambda_2}\omega_3^2 = \frac{(a^2 + b^2 - b^2 - c^2)(a^2 + b^2 - c^2 - a^2)}{(b^2 + c^2)(c^2 + a^2)}\omega_3^2$$
$$= \frac{(a^2 - c^2)(b^2 - c^2)}{(a^2 + c^2)(b^2 + c^2)}\omega_3^2 \qquad \text{(xvii)}$$

Putting in the given numbers, we find $\Omega = 0.968\omega_3 = 174$ rpm. If the book is spinning about the longest (x) axis we have only to swap λ_1 and λ_3, and we find $\Omega = 0.614\omega_3 = 111$ rpm.

10.44 ⋆⋆ Because $\lambda_1 = \lambda_2$, Euler's equations (10.88) simplify. In particular, the third equation reads $\lambda_3\dot{\omega}_3 = \Gamma$, which is easily solved to give $\omega_3 = \omega_{30}(1 + 2\beta t)$, where ω_{30} is the initial value of ω_3 and the constant $\beta = \Gamma/(2\lambda_3\omega_{30})$. That is, the spin about the symmetry axis accelerates linearly.

The first two Euler equations (10.88) now become (remember $\lambda_1 = \lambda_2$)

$$\dot{\omega}_1 = -\frac{\lambda_3 - \lambda_1}{\lambda_1}\omega_3\omega_2 = -\Omega(1 + 2\beta t)\omega_2 \quad \text{and} \quad \dot{\omega}_2 = +\frac{\lambda_3 - \lambda_1}{\lambda_1}\omega_3\omega_1 = +\Omega(1 + 2\beta t)\omega_1$$

where the constant $\Omega = \omega_{30}(\lambda_3 - \lambda_1)/\lambda_1$. Setting $\omega_1 + i\omega_2 = \eta$, we can combine these two equations as $\dot{\eta} = i\Omega(1 + 2\beta t)\eta$, which can be solved by separation of variables to give $\eta = \omega_{10}e^{i\Omega(t+\beta t^2)}$, or

$$\omega_1 = \omega_{10}\cos\Omega(t + \beta t^2) \quad \text{and} \quad \omega_2 = \omega_{10}\sin\Omega(t + \beta t^2).$$

Thus, while ω_3 accelerates, the component of $\boldsymbol{\omega}$ in the xy plane precesses with a fixed magnitude ω_{10} but at an increasing rate.

10.45 ⋆⋆ **(a)** From Equation (10.93) the rate of precession of $\boldsymbol{\omega}$ about the earth's axis $\mathbf{e}_3$ is $\Omega_b = \omega_3(\lambda_1 - \lambda_3)/\lambda_1 = 0.00327\omega_3$. The period of this precession is

$$\tau_b = \frac{2\pi}{\Omega_b} = \frac{1}{0.00327}\frac{2\pi}{\omega_3} = 306 \text{ days}$$

because $2\pi/\omega_3 = 1$ day. This is very nearly, but not quite, the claimed 305 days. The discrepancy is because $2\pi/\omega_3$ is actually 1 sidereal day (the time for one rotation of the earth relative to the stars), and a sidereal day is less than a solar day (what we normally consider to be a day) by about one part in 365. Thus 306 sidereal days are equal to about 305 solar days.

(b) From Fig.10.9 we see that $\tan\alpha = \omega_o/\omega_3$. Since α is tiny ($\alpha = 0.2$ arcseconds $\approx 10^{-6}$ rad), this means that $\omega_o \ll \omega_3$ and hence $\omega = |\boldsymbol{\omega}| \approx \omega_3$. Similarly from (10.95), $L = |\mathbf{L}| \approx L_3 = \lambda_3\omega_3$. Therefore the rate of precession of $\boldsymbol{\omega}$ in the space frame, as given by Eq.(10.96), is $\Omega_s = L/\lambda_1 \approx \lambda_3\omega_3/\lambda_1 \approx \omega_3$, and the corresponding period is $\tau_s = 2\pi/\Omega_s \approx 2\pi/\omega_3 = 1$ day.