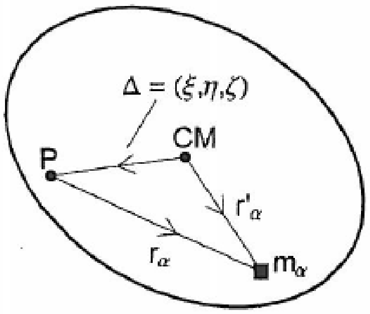
Homework-3: Problem 10.22, 10.24, 10.29, 10.35, 10.36, 10.37, 10.39

10.22 ⋆⋆ **(a)** From (10.37), $I_{xx} = \sum m_\alpha(y_\alpha^2 + z_\alpha^2) = m\,(\sum y_\alpha^2 + \sum z_\alpha^2)$. In the first sum in the last expression, four of the points lie in the plane $y = 0$, while the other four have $y_\alpha = a$; thus this first sum is $4a^2$. The same applies to the second sum, and we conclude that $I_{xx} = 8ma^2$. The other two diagonal elements are clearly the same. Similarly, from (10.38), $I_{xy} = -m \sum x_\alpha y_\alpha$. In this sum, four of the points lie in the plane $x = 0$, and of the remaining four points, two lie in the plane $y = 0$. This leaves two points, both with $x_\alpha = y_\alpha = a$. Thus $I_{xy} = -2ma^2$. All the remaining off-diagonal elements are the same, and the inertia tensor is as shown on the left below.

$$\mathbf{I}(\text{part a}) = ma^2 \begin{bmatrix} 8 & -2 & -2 \\ -2 & 8 & -2 \\ -2 & -2 & 8 \end{bmatrix} \qquad \text{and} \qquad \mathbf{I}(\text{part b}) = ma^2 \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(b) As in part (a), $I_{xx} = \sum m_\alpha(y_\alpha^2 + z_\alpha^2) = m\,(\sum y_\alpha^2 + \sum z_\alpha^2)$, but now all eight terms in both sums are the same and equal to $(a/2)^2$. Therefore $I_{xx} = 4ma^2 = I_{yy} = I_{zz}$. Because the body has reflection symmetry in all three coordinate planes, all of the off-diagonal elements are zero, and the inertia tensor is as shown above right.

10.24 ⋆⋆ **(a)** For rotation about P, the moment of inertia $I_{xx} = \sum m_\alpha(y_\alpha^2 + z_\alpha^2)$. From the picture, you can see that $\mathbf{r}_\alpha = \mathbf{r}'_\alpha - \boldsymbol{\Delta}$, so that $x_\alpha = x'_\alpha - \xi$ and so on. Therefore



$$\begin{aligned} I_{xx} &= \sum m_\alpha[(y'_\alpha - \eta)^2 + (z'_\alpha - \zeta)^2] \\ &= \sum m_\alpha({y'_\alpha}^2 + {z'_\alpha}^2) + \sum m_\alpha(\eta^2 + \zeta^2) \\ &\quad - 2\eta \sum m_\alpha y'_\alpha - 2\zeta \sum m_\alpha z'_\alpha \,. \end{aligned}$$

The first sum on the second line is just I_{xx}^{cm}. The second is $M(\eta^2 + \zeta^2)$, and the last two are zero by (10.7). Thus

$$I_{xx} = I_{xx}^{\text{cm}} + M(\eta^2 + \zeta^2) \tag{iv}$$

as claimed. The other two diagonal elements work the same way, as do the six off-diagonal terms; for instance,

$$I_{yz} = I_{yz}^{\text{cm}} - M\eta\zeta. \tag{v}$$

(b) In Example 10.2(b) we found $\mathbf{I}^{\text{cm}}$ for a cube in (10.52), which gives

$$I_{xx}^{\text{cm}} = \tfrac{1}{6}Ma^2 \qquad \text{and} \qquad I_{yz}^{\text{cm}} = 0.$$

In part (a) of the same example, we found $\mathbf{I}$ for for the same cube rotating about a corner, which is displaced from the CM by $\boldsymbol{\Delta} = (-a/2, -a/2, -a/2)$. There we found in (10.49)

$$I_{xx} = \tfrac{2}{3}Ma^2 = \tfrac{1}{6}Ma^2 + 2M(-a/2)^2 \qquad \text{and} \qquad I_{yz} = -\tfrac{1}{4}Ma^2 = 0 - M(-a/2)(-a/2).$$

As you can easily see these are precisely the relations (iv) and (v) with $\eta = \zeta = -a/2$.

10.29 $\star$ That Ox is a principal axis means that if $\boldsymbol{\omega}$ is along Ox then $\mathbf{L}$ is also along Ox. If we think in terms of matrices, this says that if $\boldsymbol{\omega}$ is a column with entries $\omega, 0, 0$, then $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$ is a column whose second and third entries are also zero. This requires that $I_{yx} = I_{zx} = 0$. Similarly, that Oy and Oz are principal axes requires that $I_{xy} = I_{zy} = 0$ and $I_{xz} = I_{yz} = 0$. This leaves $\mathbf{I}$ with I_{xx}, I_{yy}, and I_{zz} down the diagonal and zeroes everywhere else.

10.35 $\star\star$ (a)

$$I_{xx} = \sum m_\alpha(y_\alpha^2 + z_\alpha^2) = 0 + 2m(a^2 + a^2) + 3m(a^2 + a^2) = 10ma^2$$

$$I_{yy} = \sum m_\alpha(z_\alpha^2 + x_\alpha^2) = m(a^2) + 2m(a^2) + 3m(a^2) = 6ma^2 = I_{zz}$$

$$I_{yz} = -\sum m_\alpha(y_\alpha z_\alpha) = 0 - 2m(a^2) - 3m(-a^2) = ma^2$$

$$I_{zx} = -\sum m_\alpha(z_\alpha x_\alpha) = 0 + 0 + 0 = 0 = I_{xy}$$

Therefore

$$\mathbf{I} = ma^2 \begin{bmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{bmatrix}$$

(b) For convenience write the eigenvalues of $\mathbf{I}$ as $\lambda = ma^2\lambda'$. Then

$$(\mathbf{I} - \lambda\mathbf{1}) = ma^2 \begin{bmatrix} 10-\lambda' & 0 & 0 \\ 0 & 6-\lambda' & 1 \\ 0 & 1 & 6-\lambda' \end{bmatrix}. \qquad \text{(xiii)}$$

This has determinant $\det(\mathbf{I} - \lambda\mathbf{1}) = (ma^2)^3(10-\lambda')(7-\lambda')(5-\lambda')$. Therefore the three eigenvalues (that is, the three principal moments) are

$$\lambda_1 = 10ma^2, \quad \lambda_2 = 7ma^2, \quad \text{and} \quad \lambda_3 = 5ma^2$$

To find the corresponding principal axes, we must substitute λ_1, λ_2, and λ_3 in turn into (xiii) and solve the equation $(\mathbf{I} - \lambda\mathbf{1})\mathbf{a} = 0$. For the first eigenvalue, this gives three equations, $0 = 0$, $-4a_2 + a_3 = 0$, and $a_2 - 4a_3 = 0$. Therefore, $a_2 = a_3 = 0$ and $\mathbf{a} = (a_1, 0, 0)$. Thus for a unit vector in the direction of the first principal axis we can take $\mathbf{e}_1 = (1, 0, 0)$; that is, the first principal axis is the x axis. The other two principal axis are found in the same way to be

$$\mathbf{e}_2 = \tfrac{1}{\sqrt{2}}(0, 1, 1) \quad \text{and} \quad \mathbf{e}_3 = \tfrac{1}{\sqrt{2}}(0, 1, -1).$$

In this case, the second axis points toward the mass $2m$, and the third toward the mass $3m$.

10.36 $\star\star$ **(a)** The three masses are equal, $m_1 = m_2 = m_3 = m$ and their positions are

$$\mathbf{r}_1 = a(1,0,0), \qquad \mathbf{r}_2 = a(0,1,2), \qquad \text{and} \qquad \mathbf{r}_3 = a(0,2,1).$$

Therefore

$$\left.\begin{aligned}
I_{xx} &= \textstyle\sum m_\alpha(y_\alpha^2 + z_\alpha^2) = ma^2(0+5+5) = 10ma^2 \\
I_{yy} &= \textstyle\sum m_\alpha(x_\alpha^2 + z_\alpha^2) = ma^2(1+4+1) = 6ma^2 \\
I_{zz} &= \textstyle\sum m_\alpha(x_\alpha^2 + y_\alpha^2) = ma^2(1+1+4) = 6ma^2 \\
I_{xy} &= -\textstyle\sum m_\alpha x_\alpha y_\alpha = -ma^2(0+0+0) = 0 \\
I_{xz} &= -\textstyle\sum m_\alpha x_\alpha z_\alpha = -ma^2(0+0+0) = 0 \\
I_{yz} &= -\textstyle\sum m_\alpha y_\alpha z_\alpha = -ma^2(0+2+2) = -4ma^2
\end{aligned}\right\} \text{ or } \mathbf{I} = 2ma^2 \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{bmatrix}$$

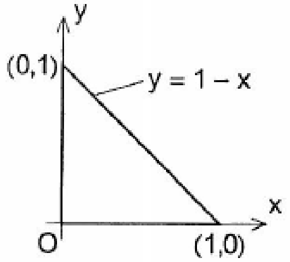
(b) As you can check, the characteristic equation is

$$\det(\mathbf{I} - \lambda\mathbf{1}) = (10ma^2 - \lambda)^2(2ma^2 - \lambda) = 0$$

Therefore, the principal moments are $\lambda_1 = \lambda_2 = 10ma^2$ and $\lambda_3 = 2ma^2$. If we set $\lambda = 10ma^2$, the equation $(\mathbf{I} - \lambda\mathbf{1})\omega = 0$ yields three equations, $0 = 0$, $\omega_2 + \omega_3 = 0$, and $\omega_2 + \omega_3 = 0$, of which only one is independent. Thus there are two independent eigenvectors with $\lambda = 10ma^2$, which we can take to be $\mathbf{e}_1 = (1,0,0)$ and $\mathbf{e}_2 = (0,1,-1)/\sqrt{2}$ or any other two perpendicular directions in the plane of these two. If we set $\lambda = 2ma^2$, the equation $(\mathbf{I} - \lambda\mathbf{1})\boldsymbol{\omega} = 0$ yields three equations, $\omega_1 = 0$, $\omega_2 - \omega_3 = 0$, and $-\omega_2 + \omega_3 = 0$. There is just one independent eigenvector with $\lambda = 2ma^2$, which we can take to be $\mathbf{e}_3 = (0,1,1)/\sqrt{2}$.

10.37 $\star\star\star$ **(a)** Since all mass is confined to the plane $z = 0$,

$$I_{xx} = \int \sigma y^2 dA = \sigma \int_0^1 dx \int_0^{1-x} y^2 dy = \sigma/12 = 2.$$



I_{yy} is the same, and from Problem 10.23, we know that

$$I_{zz} = I_{xx} + I_{yy} = 4.$$

I_{xz} and I_{yz} are both zero, and

$$I_{xy} = -\int \sigma xy\, dA = -\sigma \int_0^1 x\, dx \int_0^{1-x} y\, dy = -1.$$

Therefore

$$\mathbf{I} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(b) The characteristic equation is $\det(\mathbf{I} - \lambda\mathbf{1}) = (1-\lambda)(3-\lambda)(4-\lambda) = 0$, so the principal moments are $\lambda_1 = 1$, $\lambda_2 = 3$, and $\lambda_3 = 4$. If we set $\lambda = \lambda_1 = 1$, the equation $(\mathbf{I} - \lambda\mathbf{1})\boldsymbol{\omega} = 0$ implies the three equations $\omega_1 - \omega_2 = 0$, $\omega_1 - \omega_2 = 0$, and $\omega_3 = 0$; thus, the corresponding principal direction is $\mathbf{e}_1 = (1,1,0)/\sqrt{2}$. Setting $\lambda = \lambda_2$ and $\lambda = \lambda_3$ in turn, we can similarly find the other two principal directions to be $\mathbf{e}_2 = (1,-1,0)/\sqrt{2}$ and $\mathbf{e}_3 = (0,0,1)$.

10.39 $\star$ According to Eq.(10.83), the rate of precession is $\Omega = MgR/(\lambda_3\omega)$, where R is the distance from the tip to the CM of the cone, $R = \frac{3}{4}h$, and λ_3 is the moment of inertia about the cone's axis, $\lambda_3 = \frac{3}{10}Mr^2$ [Eq.(10.59)]. Therefore $\Omega = 5gh/(2r^2\omega) = 21$ rad/s ≈ 200 rpm.