Homework-3: Problem 8.29, 10.1, 10.5, 10.9, 10.10, 10.13, 10.16

8.29 ⋆⋆ When the mass of the sun is suddenly halved, the earth's PE is immediately halved, $U = \frac{1}{2}U_o$. On the other hand, the KE is unchanged, $T = T_o$. Therefore, the total energy becomes $E = T + U = T_o + \frac{1}{2}U_o = 0$, because $T_o = -\frac{1}{2}U_o$ in a circular orbit (virial theorem). Since the final orbit has $E = 0$, it is a parabola, and the earth would eventually escape from the sun.

10.1 ⋆ From (10.4), $\mathbf{r}'_\alpha = \mathbf{r}_\alpha - \mathbf{R}$, so

$$\sum m_\alpha \mathbf{r}'_\alpha = \sum m_\alpha \mathbf{r}_\alpha - \sum m_\alpha \mathbf{R}.$$

Now, by the definition (10.1) of the CM position, the first sum on the right is just $M\mathbf{R}$, and, by factoring the $\mathbf{R}$ from the second sum, you can see that the second term is the same. Therefore, the two terms on the right cancel, and the sum on the left is zero.

10.5 ⋆⋆ The x and y coordinates are easy: For instance, $X = (1/M)\int \varrho x dV$. Since every point (x, y, z) in the hemisphere can be paired with a point $(-x, y, z)$ and the contributions from these two points exactly cancel, we conclude that $X = 0$. Similarly $Y = 0$. Finally

$$Z = \frac{1}{M}\int \varrho z dV = \frac{\varrho}{M}\int_0^R r^2 dr \int_0^{\pi/2} \sin\theta d\theta \int_0^{2\pi} d\phi\, r\cos\theta$$
$$= \frac{1}{V}\int_0^R r^3 dr \int_0^{\pi/2} \sin\theta\cos\theta d\theta \int_0^{2\pi} d\phi = \frac{3}{8}R$$

where, in passing to the second line, I used the fact that $V = \frac{2}{3}\pi R^3$.

10.9 ⋆ Let us put the cylinder with its axis on the z axis and use cylindrical polar coordinates. The density is $\varrho = M/V$ where $V = \pi R^2 h$, and the element of volume is $dV = \rho\, d\rho\, d\phi\, dz$. Thus

$$I_{zz} = \int \varrho\rho^2 dV = \frac{M}{V}\int_0^R \rho^3 d\rho \int_0^{2\pi} d\phi \int_0^h dz = \frac{M}{\pi R^2 h}\cdot\frac{R^4}{4}\cdot 2\pi\cdot h = \tfrac{1}{2}MR^2.$$

The products of inertia I_{xz} and I_{yz} are zero because the cylinder is axially symmetric about the z axis.

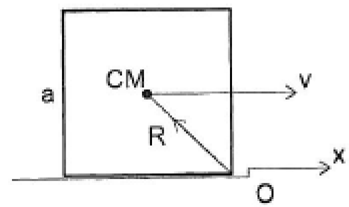
10.10 ⋆ **(a)** Since $\mu = M/L$, $I = \int_0^L \mu x^2 dx = \frac{M}{L}\left[x^3/3\right]_0^L = \frac{1}{3}ML^2$.

(b) $I = \int_{-L/2}^{L/2} \mu x^2 dx = \frac{M}{L}\left[x^3/3\right]_{-L/2}^{L/2} = \frac{1}{12}ML^2$.

10.13 ⋆⋆ **(a)** The equation $\dot{L}_z = \Gamma_z$ implies that $I\ddot{\phi} = -mga\sin\phi \approx -mga\phi$. This implies SHM with angular frequency $\omega = \sqrt{mga/I}$ and hence period $\tau = 2\pi\sqrt{I/(mga)}$.

(b) The period of a simple pendulum is $\omega = \sqrt{g/L}$ and these two periods are equal if $L = I/(ma)$.

10.16 ⋆⋆ **(a)** The moment of inertia of the cube about any edge is worked out in Example 10.2 and is given by (10.47) as $\frac{2}{3}Ma^2$. During the collision, kinetic energy will be lost (the collision is inevitably inelastic), but the angular momentum L_y about the edge of the step is conserved. (I take the x direction as that of the incident velocity, as shown, and y into the page.) Just before the collision the angular momentum is $\sum m_\alpha \mathbf{r}_\alpha \times \mathbf{v} = M\mathbf{R} \times \mathbf{v}$, so that $L_y = Mav/2$. Just after the collision, the cube is rotating about the edge of the step and $L_y = I_{yy}\omega_o = \frac{2}{3}Ma^2\omega_o$. Equating these two expressions for L_y, we find that $\omega_o = 3v/(4a)$.

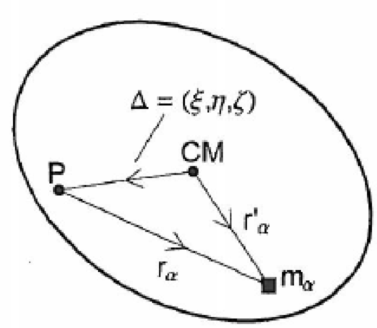


(b) If the initial speed is small, the cube's rotational motion about O will stop before the CM has passed the step, and the cube will fall backward. If v is big enough, the CM will pass the step and the cube will roll forward. At the critical speed that divides these possibilities, the CM will just come to rest vertically above O. Since mechanical energy is conserved in the rotational phase of motion, this critical speed is determined by the condition

$$\tfrac{1}{2}I_{yy}{\omega_o}^2 + Mga/2 = Mga/\sqrt{2}.$$

(The height of the CM above O is $a/2$ initially, and $a/\sqrt{2}$ when the CM is vertically above O.) Substituting for ω_o from part (a), we can solve for v and find $v_{\text{crit}} = [8(\sqrt{2}-1)ga/3]^{1/2}$.

10.24 ⋆⋆ **(a)** For rotation about P, the moment of inertia $I_{xx} = \sum m_\alpha(y_\alpha^2 + z_\alpha^2)$. From the picture, you can see that $\mathbf{r}_\alpha = \mathbf{r}'_\alpha - \mathbf{\Delta}$, so that $x_\alpha = x'_\alpha - \xi$ and so on. Therefore



$$\begin{aligned} I_{xx} &= \sum m_\alpha[(y'_\alpha - \eta)^2 + (z'_\alpha - \zeta)^2] \\ &= \sum m_\alpha({y'_\alpha}^2 + {z'_\alpha}^2) + \sum m_\alpha(\eta^2 + \zeta^2) \\ &\quad - 2\eta \sum m_\alpha y'_\alpha - 2\zeta \sum m_\alpha z'_\alpha . \end{aligned}$$

The first sum on the second line is just I_{xx}^{cm}. The second is $M(\eta^2 + \zeta^2)$, and the last two are zero by (10.7). Thus

$$I_{xx} = I_{xx}^{\text{cm}} + M(\eta^2 + \zeta^2) \qquad \text{(iv)}$$

as claimed. The other two diagonal elements work the same way, as do the six off-diagonal terms; for instance,

$$I_{yz} = I_{yz}^{\text{cm}} - M\eta\zeta. \qquad \text{(v)}$$

(b) In Example 10.2(b) we found $\mathbf{I}^{\text{cm}}$ for a cube in (10.52), which gives

$$I_{xx}^{\text{cm}} = \tfrac{1}{6}Ma^2 \qquad \text{and} \qquad I_{yz}^{\text{cm}} = 0.$$

In part (a) of the same example, we found $\mathbf{I}$ for for the same cube rotating about a corner, which is displaced from the CM by $\mathbf{\Delta} = (-a/2, -a/2, -a/2)$. There we found in (10.49)

$$I_{xx} = \tfrac{2}{3}Ma^2 = \tfrac{1}{6}Ma^2 + 2M(-a/2)^2 \qquad \text{and} \qquad I_{yz} = -\tfrac{1}{4}Ma^2 = 0 - M(-a/2)(-a/2).$$

As you can easily see these are precisely the relations (iv) and (v) with $\eta = \zeta = -a/2$.