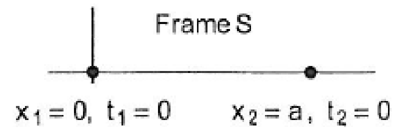
Homework #10, Problem: 15.17, 15.23, 15.27, 15.39, 15, 40, 15.42, 15.43

15.17 ⋆



(a) Frame $\mathcal{S}'$ has velocity V relative to $\mathcal{S}$.
Therefore

$$t_1' = \gamma(t_1 - \beta x_1/c) = 0 \text{ and } t_2' = \gamma(t_2 - \beta x_2/c) = -\gamma\beta a/c.$$

(b) Frame $\mathcal{S}''$ has velocity $-V$ relative to $\mathcal{S}$.
Therefore

$$t_1' = \gamma(t_1 + \beta x_1/c) = 0 \text{ and } t_2' = \gamma(t_2 + \beta x_2/c) = \gamma\beta a/c.$$

In frame $\mathcal{S}$ the two events are simultaneous. In $\mathcal{S}'$ event 1 is later than 2, and in $\mathcal{S}''$ event 1 is earlier than 2.

15.23 ⋆ If we let $\mathcal{S}'$ denote the rest frame of the left rocket, then the velocity of $\mathcal{S}'$ relative to $\mathcal{S}$ is $V = 0.9c$. The velocity of the right rocket relative to $\mathcal{S}$ has $v_x = -0.9c$ and, relative to $\mathcal{S}'$,

$$v_x' = \frac{v_x - V}{1 - v_x V/c^2} = \frac{-0.9c - 0.9c}{1 + 0.9 \times 0.9} = -\frac{1.8c}{1.81} = -0.994c.$$

15.27 ⋆⋆ Consider first the final x coordinate. This is $x'' = \gamma_2(x' - V_2 t')$, where γ_2 is the γ factor corresponding to the second velocity V_2. The coordinates x' and t' are given by the first Lorentz transformation (velocity V_1), and substituting these values we find that

$$\begin{aligned} x'' &= \gamma_1\gamma_2[(x - V_1 t) - V_2(t - V_1 x/c^2)] \\ &= \gamma_1\gamma_2[x(1 + V_1V_2/c^2) - (V_1 + V_2)t] \\ &= \gamma_1\gamma_2\left(1 + \frac{V_1V_2}{c^2}\right)\left[x - \left(\frac{V_1 + V_2}{1 + V_1V_2/c^2}\right)t\right] \\ &= \gamma_1\gamma_2(1 + \beta_1\beta_2)\big[x - Vt\big] \qquad \text{(viii)} \end{aligned}$$

In the last line I have made two substitutions. In the interests of tidiness, I have replaced V_1/c by β_1 and V_2/c by β_2. More important, I have recognized that the coefficient of t in the previous line is the relativistic "sum"

$$V = \frac{V_1 + V_2}{1 + V_1V_2/c^2}$$

of the two separate velocities V_1 and V_2. The form (viii) for x'' is very close to the standard Lorentz transformation for the single velocity V. All that remains to be shown is that the product to the left of the square bracket is equal to the γ factor for the velocity V. To show this, let's evaluate the latter:

$$\begin{aligned} \gamma_V = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1 - (\beta_1+\beta_2)^2/(1+\beta_1\beta_2)^2}} &= \frac{(1+\beta_1\beta_2)}{\sqrt{(1+\beta_1\beta_2)^2 - (\beta_1+\beta_2)^2}} \\ &= \frac{(1+\beta_1\beta_2)}{\sqrt{(1-\beta_1{}^2)(1-\beta_2{}^2)}}\,. \end{aligned}$$

Comparing with (viii) we see that indeed $x'' = \gamma_V(x - Vt)$; that is, the two successive Lorentz transformations with velocities V_1 and V_2 produce the same effect as a single Lorentz transformation with velocity V equal to the relativistic "sum" of V_1 and V_2. The transformations of y and z are trivial (for example, $y'' = y' = y$) and that of the time works just the same as that of x.

15.39 ⋆ If $x \cdot x < 0$ in frame $\mathcal{S}$, then $x' \cdot x' < 0$ in any other frame $\mathcal{S}'$, since $x \cdot x$ has the same value in all frames. The condition $x \cdot x = \mathbf{x}^2 - x_4^2 < 0$ implies that $|\mathbf{x}| < |x_4|$. Now suppose, in addition, that $x_4 < 0$ in frame $\mathcal{S}$ and let $\mathcal{S}'$ be obtained from $\mathcal{S}$ by a standard boost. Then

$$x_4' = \gamma(x_4 - \beta x_1) < 0$$

since $|\beta| < 1$ and $|x_1| \le |\mathbf{x}| < |x_4|$. That is, $x_4' < 0$ in $\mathcal{S}'$. Since x_4 is unchanged by any rotation, the same conclusion holds in all frames $\mathcal{S}'$.

15.40 ⋆ A point x in space-time lies on the forward light cone if and only if $x \cdot x = 0$ and $x_4 > 0$. We have to show that if these two conditions hold in a frame $\mathcal{S}$, they automatically hold in any other frame $\mathcal{S}'$. This is certainly true of the condition $x \cdot x = 0$ since $x \cdot x$ is Lorentz invariant. To check the second condition, note that because $x \cdot x = \mathbf{x}^2 - x_4^2 = 0$, it follows that $|\mathbf{x}| = |x_4|$. Now suppose $x_4 > 0$ (in frame $\mathcal{S}$) and let's consider a frame $\mathcal{S}'$ related to $\mathcal{S}$ by the standard boost. In $\mathcal{S}'$

$$x_4' = \gamma(x_4 - \beta x_1) > 0$$

because $|\beta| < 1$ and $|x_1| \leq |\mathbf{x}| = |x_4|$. That is, $x_4' > 0$ in S'. Since x_4 is unchanged by any rotation, the same conclusion holds in all frames S'.

15.42 ⋆ That x is time-like means that $|\mathbf{x}| < |x_4|$. That $x{\cdot}y = 0$ means that $\mathbf{x}\cdot\mathbf{y} - x_4 y_4 = 0$ or equivalently

$$|\mathbf{x}|\,|\mathbf{y}|\cos\theta = x_4 y_4.$$

Since $|\mathbf{x}| < |x_4|$, this implies that $|\mathbf{y}|\,|\cos\theta| > |y_4|$ which guarantees that $|\mathbf{y}| > |y_4|$ and hence that y is space-like. (In the special case that $|\mathbf{x}| = 0$, it is clear that y_4 must be zero, and again y is space-like.)

15.43 ⋆ (a) Suppose that the body moves from $\mathbf{x}$ to $\mathbf{x} + d\mathbf{x}$ as the time advances from t to $t + dt$. Let dx denote the four-vector displacement $dx = (d\mathbf{x}, c\,dt) = (\mathbf{v}, c)dt$ and consider the following two equivalent statments:

$$|\mathbf{v}| < c \iff dx^2 < 0. \tag{ix}$$

Since dx^2 is Lorentz invariant, the second condition, if true in one frame, must be true in all frames. The same must therefore apply to the first; that is, if $|\mathbf{v}| < c$ in one frame, then $|\mathbf{v}| < c$ in all frames.

(b) The argument for a signal with speed c is the same except that the two conditions (ix) are replaced by $|\mathbf{v}| = c \iff dx^2 = 0$.