Homework #1, Problem: 8.2, 8.6, 8.7, 8.12, 8.18, 8.34, 8.35

8.2 ★★ **(a)** The Lagrangian is $\mathcal{L} = T - U$ or

$$\mathcal{L} = \tfrac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \tfrac{1}{2}m_2\dot{\mathbf{r}}_2^2 - \left[m_1 g z_1 + m_2 g z_2 + U(r)\right]$$
$$= \left[\tfrac{1}{2}M\dot{\mathbf{R}}^2 - MgZ\right] + \left[\tfrac{1}{2}\mu\dot{\mathbf{r}}^2 - U(\mathbf{r})\right] = \mathcal{L}_{\rm cm} + \mathcal{L}_{\rm rel},$$

where I have chosen rectangular coordinates with z measured vertically up and Z is the z coordinate of the CM position, $Z = (m_1 z_1 + m_2 z_2)/M$.

(b) The Lagrange equations for the three components of $\mathbf{R}$ are

$$M\ddot{X} = 0, \quad M\ddot{Y} = 0, \quad M\ddot{Z} = -g,$$

so the CM moves just like a projectile of mass M. The Lagrange equations for the relative coordinates are

$$\mu\ddot{\mathbf{r}} = -\nabla_{\mathbf{r}}\, U(r)$$

where $\nabla_{\mathbf{r}}$ denotes the gradient with respect to the relative coordinates. This last equation is precisely Newton's second law for the motion of a single particle of mass μ, position $\mathbf{r}$, and potential energy $U(r)$.

8.6 ★ In the CM frame, with the origin fixed at the CM, we know that $\mathbf{p}_2 = -\mathbf{p}_1$ and that $\mathbf{r}_2 = -(m_1/m_2)\mathbf{r}_1$. Therefore

$$\mathbf{L} = \mathbf{r}_1 \times \mathbf{p}_1 + \mathbf{r}_2 \times \mathbf{p}_2 = \left(1 + \frac{m_1}{m_2}\right)\mathbf{r}_1 \times \mathbf{p}_1 = \frac{M}{m_2}\boldsymbol{\ell}_1$$

and $\boldsymbol{\ell}_1 = (m_2/M)\mathbf{L}$. Similary with $\boldsymbol{\ell}_2$.

8.7 ★★ **(a)** Newton's second law in the form $m\mathbf{a} = \mathbf{F}$, with $\mathbf{a}$ equal to the centripetal acceleration (magnitude $\omega^2 r$), implies that $m_1\omega^2 r = Gm_1m_2/r^2$. Substituting $\omega = 2\pi/\tau$ and solving for the period τ, we find $\tau = 2\pi r^{3/2}/\sqrt{Gm_2}$. (Note how the mass m_1 cancels out.)

(b) The equation of motion, $\mu\ddot{\mathbf{r}} = \mathbf{F}$, for the relative motion (with μ equal to the *reduced* mass m_1m_2/M) implies that $\mu\omega^2 r = Gm_1m_2/r^2$. This is the same as before, except that the mass on the left is now the reduced mass μ instead of m_1. This no longer cancels the factor m_1 on the right, so the final answer contains m_1m_2/μ instead of m_2:

$$\tau = \frac{2\pi r^{3/2}}{\sqrt{Gm_1m_2/\mu}} = \frac{2\pi r^{3/2}}{\sqrt{GM}}. \qquad \text{(i)}$$

If $m_2 \gg m_1$, then $M \approx m_2$ and (i) approaches the answer of part (a).

(c) If $m_1 = m_2$, then $M = 2m_2$, and the period (i) is $1/\sqrt{2}$ times that in part (a), or, for the case at hand, 0.71 years.

8.12 ⋆⋆ (a) According to Eq.(8.29), $\mu\ddot{r} = -dU_{\rm eff}/dr$. Therefore, the planet can orbit at a fixed radius if and only if $dU_{\rm eff}/dr = 0$. Since $U_{\rm eff} = -\gamma/r + \ell^2/2\mu r^2$, it follows that $dU_{\rm eff}/dr = \gamma/r^2 - \ell^2/\mu r^3$, which is zero when $r = r_{\rm o} = \ell^2/\gamma\mu$.

(b) The "equilibrium" radius $r_{\rm o}$ is stable if and only if $U_{\rm eff}$ is minimum at $r_{\rm o}$; that is, its second derivative must be positive. This derivative is

$$\left[\frac{d^2U_{\rm eff}}{dr^2}\right]_{r=r_{\rm o}} = \left[\frac{-2\gamma}{r^3} + \frac{3\ell^2}{\mu r^4}\right]_{r=r_{\rm o}} = \frac{\gamma}{r_{\rm o}^3}$$

where, in the second equality, I used the result of part (a) to write $\ell^2 = \gamma\mu r_{\rm o}$. Since this second derivative is positive, the equilibrium is stable. Near the minimum, the effective PE has the approximate form $U_{\rm eff} \approx \text{const} + \frac{1}{2}(\gamma/r_{\rm o}^3)(r - r_{\rm o})^2$. Substituting this into the equation of motion, we get $\mu\ddot{r} = -dU_{\rm eff}/dr = -(\gamma/r_{\rm o}^3)(r - r_{\rm o})$, which shows that r oscillates about $r_{\rm o}$ with angular frequency $\omega_{\rm osc} = \sqrt{\gamma/\mu r_{\rm o}^3}$, which is exactly the same as the angular velocity of the planet in its circular orbit. (To see this, set the centripetal acceleration $\omega^2 r$ equal to $F_{\rm grav}/m$.) Therefore, the period of oscillation is equal to the orbital period.

8.18 ⋆⋆ We are given the satellite's height $h_{\rm min} = 250$ km and speed $v_{\rm max} = 8500$ m/s at perigee. The distance from the earth's center is then $r_{\rm min} = R_{\rm e} + h_{\rm min} = 6650$ km. For any known satellite, we can certainly ignore the difference between the mass m and the reduced mass $\mu \approx m$. Thus the angular momentum is $\ell = mv_{\rm max}r_{\rm min}$ and the parameter c of Eq.(8.48) is $c = \ell^2/\gamma\mu = (v_{\rm max}r_{\rm min})^2/GM_{\rm e}$. (Recall that $\gamma = GM_{\rm e}m$.) Putting in the given numbers, we get $c = 7960$ km. The rest is easy: From Eq.(8.50), $r_{\rm min} = c/(1+\epsilon)$, so $\epsilon = (c - r_{\rm min})/r_{\rm min} = 0.197$. Similarly, from Eq.(8.50) $r_{\rm max} = c/(1-\epsilon) = 9910$ km, so $h_{\rm max} = r_{\rm max} - R_{\rm e} = 3510$ km.

8.34 ⋆⋆ If we use the notation of Example 8.6, $R_1 = 1$ AU and $R_3 = 30$ AU. Therefore the semi-major axis of the transfer orbit is $a_2 = (R_1 + R_3)/2 = 15.5$ AU and the period of the transfer orbit is $\tau_2 = \tau_{\rm e}(a_2/a_{\rm e})^{3/2} = \tau_{\rm e}(15.5)^{3/2}$. The time for the transfer is half of this period, namely $\frac{1}{2}\tau_2 = \frac{1}{2}\tau_{\rm e}(15.5)^{3/2} = 30.5$ years.

8.35 ⋆⋆⋆ The simplest way to do this problem is to imagine Example 8.6 run backwards and to invert the thrust factors found there. The first thrust comes at the point P' of Figure 8.13, and its thrust factor is [compare (8.74)] $\lambda' = \sqrt{2R_1/(R_1 + R_3)} = \sqrt{2/5} = 0.63$, where R_3 denotes the larger radius and R_1 the smaller, as in Figure 8.13. Similarly, the second thrust comes at P and its thrust factor is [compare (8.73)] $\lambda = \sqrt{(R_1 + R_3)/2R_3} = \sqrt{5/8} = 0.79$. The speed changes during the transfer orbit, and, by conservation of angular momentum, $v(\text{at } P) = v(\text{at } P')R_3/R_1$. Therefore

$$v_{\rm fin} = \lambda \cdot \frac{R_3}{R_1} \cdot \lambda' \cdot v_{\rm in} = \sqrt{\frac{R_1 + R_3}{2R_3}} \cdot \frac{R_3}{R_1} \cdot \sqrt{\frac{2R_1}{(R_1 + R_3)}} \cdot v_{\rm in} = \sqrt{\frac{R_3}{R_1}} \cdot v_{\rm in} = 2v_{\rm in}$$