Physics 201 Lab 3: Measurement of the local gravitational field of the Earth

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Theoretical Discussion

Gravity is one of the four known fundamental forces. Besides gravity, these are; the electromagnetic force, the weak force, and the strong (nuclear) force. In addition to these four known forces, there is strong evidence for the existence of a fifth force which is long-ranged, repulsive, and depends on mass in some manner—a sort of anti-gravity. The exact source and nature of this force, and even definitive proof of its existence, remain elusive. Gravity, by contrast, was the first force to be given a "correct" mathematical description. Sir Isaac Newton discovered a universal mathematical model for the gravitational force that is valid for all classical (non-quantum mechanical), non-relativistic interactions where the field is not "too large". This model was extended by Einstein in the early 20^{th} century in his General Theory of Relativity to cover the cases of relativistic velocities (velocities comparable to the speed of light) and very large masses. Interestingly, although Gravity was the first of the fundamental forces to be given a mathematical description, it is the only force for which a relativistic, quantum-mechanical model has not yet been developed. The problem of quantum gravity is one of the most theoretically challenging and important unsolved problems in physics today.

As important as a relativistic, quantum-mechanical description of gravity is for a fundamental understanding of the workings of the universe, as a practical matter Newton's universal gravitational force law is sufficient for nearly every circumstance likely to be encountered in the ordinary course of events. This model states that the force between any two masses m_1 and m_2, separated by a displacement vector $\vec{r}$ is given by

$$\vec{F_G} = -\frac{Gm_1 m_2}{r^2}\hat{r} \tag{1}$$

Like all fundamental forces, the gravitational interaction is *binary*, meaning that it always acts between pairs of objects. One can arbitrarily identify one object as the source of the force field, and the other as the object of the force field. For example, if one takes m_1 as the source of the gravitational field, then the force exerted by m_1 on m_2 is described by equation 1, with the unit vector $\hat{r}$ pointing from m_1 to m_2. The negative sign, a consequence of the attractive character of gravity, means that the force on m_2 due to m_1 acts in the direction opposite to the unit vector $\hat{r}$; i.e., the force on m_2 due to m_1 points from m_2 to m_1. Conversely, one could take m_2 as the source of the field, and m_1 its object. In this case, the unit vector $\hat{r}$ points from m_2 to m_1 and the force acts in the opposite direction, so that the force of m_2 on m_1 acts along the line separating them, in the direction from m_1 to m_2. In general terms, the unit vector $\hat{r}$ points from the source mass to the object mass, and because gravity is attractive, the force acts in the opposite direction, from the object mass to the source mass. Figure 1 depicts the directions of the forces of two massive objects on one another.

We can therefore decompose the gravitational force that a source mass m_s exerts on an object mass m_o into the product of the object mass and the gravitational field due to the source mass, so that

$$\vec{F}_{m_s \text{on} m_o} = \left[-\frac{Gm_s}{r^2}\hat{r}\right] m_o = \vec{\Gamma}_s m_o \tag{2}$$

where the gravitational field created by the source mass is given by

$$\vec{\Gamma}_s = -\frac{Gm_s}{r^2}\hat{r} \tag{3}$$

and the unit vector $\hat{r}$ points from m_s to m_o. It is in this form that the gravitational force exerted by the earth on a mass m near its surface is probably most familiar to you:

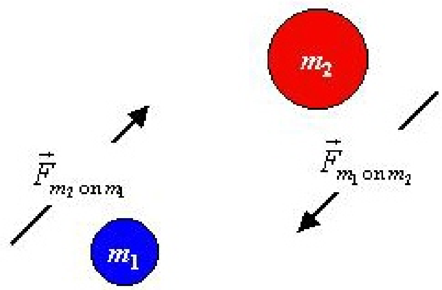

FIG. 1: Directions of gravitational forces of two massive objects on one another

$$\vec{F}_{G_E} = m\vec{\Gamma}_E = -mg\hat{y} \tag{4}$$

where the unit vector $\hat{y}$ points in the local direction of "up". This direction, of course, is just the direction of the unit vector pointing from the center of the earth to the local position of the mass m. We can then identify the gravitational field of the earth near its surface as

$$\vec{\Gamma}_E = -g\hat{y} \tag{5}$$

The numerical value of the earth's gravitational field "constant" g can be found by evaluating the magnitude of $\vec{\Gamma}$ from equation 3, using the values of the universal gravitational constant $G = 6.67 \times 10^{-11}\ \frac{\mathrm{N \cdot m^2}}{\mathrm{kg^2}}$, the mass of the earth, $M_E = 5.94 \times 10^{24}$ kg, and the earth's radius[1], $R_E = 6.37 \times 10^6$ m:

$$\begin{aligned} g &= \frac{GM_E}{R_E^2} \\ &= \frac{(6.67 \times 10^{-11})(5.94 \times 10^{24})}{(6.37 \times 10^6)^2} \\ &= 9.76\ \frac{\mathrm{m}}{\mathrm{s}^2} \end{aligned}$$

The magnitude of the earth's gravitational field is not really constant. For one thing, the density of the earth is not uniform, creating small local variations in the earth's gravitational field. In addition, g depends on r, the distance of the object mass from the center of mass of the earth. The earth is not a perfect sphere. Besides macroscopic distortions caused by its rotation, it's surface is not smooth, but is puckered with mountains and valleys. Even if the earth were a perfectly smooth sphere, it should be clear that g decreases as one moves upward from the earth's surface. One can use the Taylor series expansion[2] to show that g varies with the ratio $\delta = \frac{y}{R_E}$ according to $g \approx g(1 - 2\delta)$, where $y = (r - R_E)$ denotes the distance above or below the mean radius of the earth. The reason we can treat g as a constant for most ordinary applications is that typically, $\delta \ll 1$.

In today's lab we are going to measure the local value of g by dropping a slotted plate through a photogate timer. The timer will give us a series of measurements of the velocity v of the plate as a function of the vertical position y. Recall that we have two kinematic equations to describe the motion of an object under constant acceleration:

$$y = y_0 + v_0 t + \frac{1}{2}at^2 \tag{6}$$

and

$$v = v_0 + at \tag{7}$$

Since we do not have access to experimental information about the time t, we are going to have to use these two equations to eliminate it as a variable. Solving equation 7 for t, we get

$$t = \frac{(v - v_0)}{a}$$

Inserting this result into equation 6, we get

$$\begin{aligned} y &= y_0 + v_0\left(\frac{(v - v_0)}{a}\right) + \frac{a}{2}\left(\frac{(v - v_0)}{a}\right)^2 \\ &= y_0 + \frac{1}{a}\left(v_0 v - v_0^2 + \frac{v^2}{2} + \frac{v_0^2}{2} - v v_0\right) \\ &= y_0 + \frac{1}{2a}(v^2 - v_0^2) \end{aligned}$$

Solving for v^2, we obtain

$$v^2 = 2a(y - y_0) + v_0^2$$

and with $a = -g$, we have the desired relation between the velocity and g.

$$v^2 = -2g(y - y_0) + v_0^2 \tag{8}$$

Inspection of equation 8 reveals two facts pertinent for our analysis. The first is that only the change in vertical position, $\Delta y = (y - y_0)$ is relevant for determining g, so that we can arbitrarily choose the origin of the y-axis. The second is that the initial velocity appears only as a constant offset. If we rewrite the equation as

$$v^2 = -2g(\Delta y) + v_0^2 \tag{9}$$

it is clear that the slope m of the v^2 vs Δy curve is proportional to g, and that the square of the initial velocity, v_0^2, which is of no physical interest, appears only as the offset constant, so that we don't require *a priori* knowledge of this parameter and can, in fact, neglect it.

Procedure

A schematic diagram of the experimental setup is shown in figure 2A. The slotted bar is dropped vertically through the aperture of a photogate timing device. The timing module will be set to the *s2* functional mode. In this mode of operation, a timing cycle begins when the photogate beam is interrupted and ends upon being interrupted again. The next cycle begins when the beam is once again interrupted, and so on. The slotted bar, depicted in figure 2B, consists of a series of 1 cm wide slots, each 1 cm apart. The distance over which a timing cycle occurs is therefore 2 cm. Because a new cycle doesn't begin until the beam is again interrupted, the timing cyles themselves are spatially separated by 4 cm, as shown in the figure.

Prior to conducting the experiment, you should verify that the function mode is set to *s2*. The function mode can be altered by repeatedly pushing the function button. The device cycles through each of the functions; an LED indicates which function is activated. There is also an LED indicator that identifies the units in

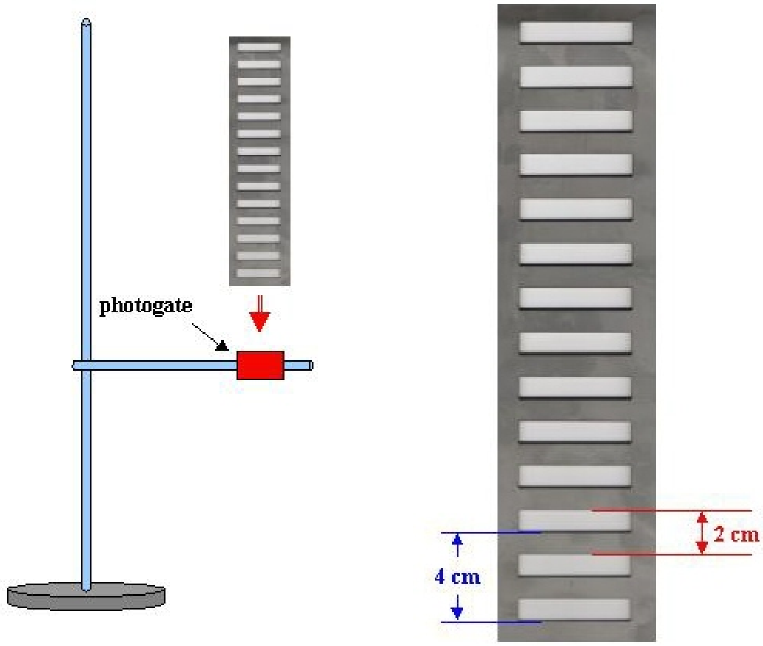

Figure 2A Figure 2B

FIG. 2: Physical setup and dimensions for the measurement of g

which the result will be output on the display panel. You should press the *clear* button, to clear out any previous results. After dropping the bar through the timing gate, press the *stop* button. The display unit will then cyclically read out the time intervals. Preceding each time interval readout, the display will indicate which time interval is about to be shown; i.e., $1^{st}, 2^{nd}, 3rd, \ldots$. You should record these times, starting from # 1 through #7. This is your raw data.

Figure 3 depicts an appropriate coordinate system for analyzing the data. The average velocity in the j^{th} time interval Δt_j is equal to[3]

$$v_j = \frac{d_t}{\Delta t_j} \tag{10}$$

where $d_t = 2$ cm, as shown in figure 2B. Because the spatial distance between the middle of each timing cycle is equal to $d_y = 4$ cm, the value of the vertical displacement for the j^{th} time interval is equal to

$$\Delta y_j = -(j d_y + \Delta y_0) \tag{11}$$

where $d_y = 4$ cm. The value of Δy_0 is arbitrary. The simplest choice is to make it equal to zero, so that

$$\Delta y_j = -j d_y \tag{12}$$

In summary, the procedure is as follows:

1. Set up the timing device so that the functional mode is *s2*. Clear any previous data.
2. Carefully drop the slotted bar vertically through the photogate timer aperture.
3. Press the *stop* button on the timing module.

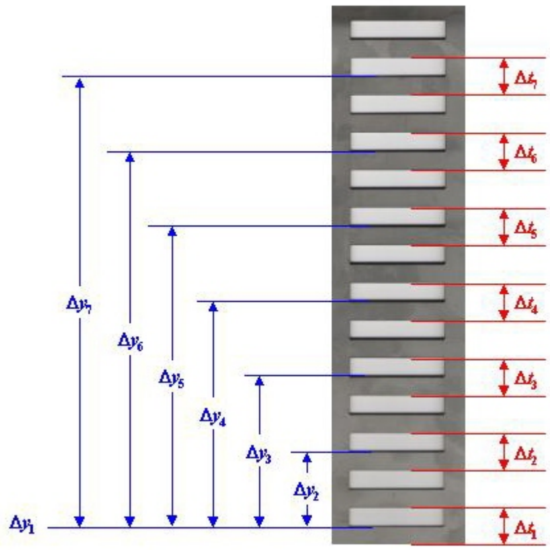

FIG. 3: Coordinate system for analyzing data from the measurement of g

4. Record the time intervals Δt_j. Pay attention to units
5. Calculate the velocities v_j using equation 10. Convert the dimensions to meters/s.
6. Calculate the displacements Δy_j using equation 12 and convert the result to meters.
7. Square each of the velocities and plot v_j^2 vs Δy_j.
8. Find the slope of this curve and determine g from $g = -\frac{m}{2}$. Report your result in m/s^2.
9. Numerically compare your result to the value $g = 9.76$ m/s^2 by calculating the fractional discrepancy.

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[1] This is the mean radius of the earth. The earth is somewhat flattened at the poles, and bulges at the equator so that, for instance, the equatorial radius is larger than the mean radius.

[2] The Taylor Series expansion allows one to expand any analytic function $f(x)$ about a point x_0 in an infinite series given by $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n f}{dx^n}(x - x_0)^n$. If we take a function of the form $f(r) = \frac{A}{r^2} = Ar^{-2}$ and apply the Taylor series expansion in a series about the point $r = R_E$, we get $f(r) \approx f(R_E) + \frac{df}{dr}\big|_{r=R_E}(r - R_E) + \frac{1}{2}\frac{d^2 f}{dr^2}\Big|_{r=R_E}(r - R_E)^2 + \cdots$. The first and second derivatives can be calculated as $\frac{df}{dr}\big|_{r=R_E} = -2A\, r^{-3}\big|_{r=R_E} = -2AR_E^{-3}$, and $\frac{d^2 f}{dr^2}\Big|_{r=R_E} = 6A\, r^{-4}\big|_{r=R_E} = 6AR_E^{-4}$, respectively, so the expansion to second order is $f(r) \approx \frac{A}{R_E^2} - 2\frac{A}{R_E^3}(r - R_E) + 3\frac{A}{R_E^4}(r - R_E)^2$, which we can re-write as $f(r) \approx \frac{A}{R_E^2}\left(1 - 2\left[\frac{(r-R_E)}{R_E}\right] + 3\left[\frac{(r-R_E)}{R_E}\right]^2\right)$. Setting $\frac{A}{R_E^2} = g$, and taking $\delta = \frac{y}{R_E}$ and $y = (r - R_E)$, we have, to second order, $\Gamma = g\left(1 - 2\delta + 3\delta^2\right)$. The expression in the text retains only the first order of this expansion.

[3] Note that this is *not* equal to the instantaneous velocity at the midpoint of the interval. This is because the slotted bar is accelerating as it falls. However, the approximation is sufficient for us to obtain the requisite accuracy.