A Modular Presentation System for the Calculus Sequence

5.2 The Definite Integral

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The Definite Integral as a Limit

Let f be a function defined on a closed interval $[a, b]$. For any partition P of $[a, b]$, let the numbers c_k be chosen arbitrarily in the subintervals $[x_{k-1}, x_k]$.

If there exists a number I such that

$$\lim_{||P||\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = I$$

no matter how P and the c_k's are chosen, then f is **integrable** on $[a, b]$ and I is the **definite integral** of f over $[a, b]$.

The Existence of Definite Integrals

Theorem: All continuous functions are integrable. That is, if a function f is continuous on an interval $[a, b]$, then its definite integral over $[a, b]$ exists.

Notation:

$$\int_a^b f(x)\, dx$$

Estimating Area Under a Function

Using mid-point rule with $n = 5$ to estimate $\int_0^1 \frac{1}{1+x^2}\, dx$.

Properties of the Definite Integral

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx.$
2. $\int_a^a f(x)dx = 0.$
3. $\int_a^b cdx = c(b-a)$, where c is a constant.
4. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx.$
5. $\int_a^b cf(x)dx = c\int_a^b f(x)dx.$
6. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$

Examle

Example If $\int_0^1) f(x)dx = 10$ and $\int_8^{10} f(x)dx = 12$, find $\int_8^{10} f(x)dx$.

Comparison Properties of the Definite Integral

1. If $f(x) \geq 0$ for all x in $[a, b]$, then $\int_a^b f(x)dx \geq 0$.
2. If $f(x) \geq g(x)$ for all x in $[a, b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.
3. If $n \leq f(x) \leq M$ for all x in $[a, b]$, then

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

Example

Example Estimate

$$\int_0^{\pi/4} \cos x \, dx$$