A Modular Presentation System for the Calculus Sequence

3.5 The Chain Rule

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The Chain Rule

The Chain Rule

If f and g are differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then F is differentiable and F' is given by the product

$$F'(x) = f'(g(x))g'(x)$$

The Chain Rule in Leibniz Notation

The Chain Rule (Leibniz Notation)

If $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Inner and Outer Functions

EXAMPLE: Find inner and outer functions for

$$y = \sin(x^2 + x)$$

EXAMPLE: Find inner and outer functions for

$$y = e^{5x}$$

Applying the Chain Rule

EXAMPLE: Find the derivative of

$$y = \sin(x^2 + x)$$

EXAMPLE: Find the derivative of

$$y = e^{5x}$$

Power Chain Rule

The Power Chain Rule

If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Applying the Power Chain Rule

EXAMPLE: Find the derivative of

$$y = (x+5)^2$$

EXAMPLE: Find the derivative of

$$y = (x^2 + 5x + 7)^{10}$$

Derivative of a^x

EXAMPLE: Noting that

$$a^x = e^{(x \ln a)}$$

find the derivative of

$$y = a^x$$

Partial Proof of Chain Rule

$$
\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\
&= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\
&= \frac{dy}{du}\frac{du}{dx}
\end{aligned}
$$

Where is the flaw in this argument?

Repeated Use of the Chain Rule

Where do I start?

Consider the function evaluation . . .

EXAMPLE: Find the derivative of

$$y = e^{\sin(5x)}$$

EXAMPLE: Find the derivative of

$$y = \sin^2(7x)$$