A Modular Presentation System for the Calculus Sequence

3.11 Linear Approximations and Differentials

Yaw Chang

Michael Freeze

Mathematics and Statistics UNC-Wilmington

Definition

If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the linearization of f at a.

Finding a Linearization

EXAMPLE: Find the linearization of

$$f(x) = \sqrt{1 + x}$$

at $x = 0$.

Finding a Second Linearization

EXAMPLE: Find the linearization of

$$f(x) = \sqrt{1 + x}$$

at $x = 3$.

Common Linear Approximations

When x is very close to 0,

$$\begin{aligned} \sin x &\approx x \\ \cos x &\approx 1 \\ \tan x &\approx x \\ (1+x)^k &\approx 1+kx \end{aligned}$$

Differentials

Definition

Let $y = f(x)$ be a differentiable function. The differential dx is an independent variable. The differential dy is $dy = f'(x)dx$.

Estimating Absolute Change

Differential Estimate of Absolute Change

Let $f(x)$ be differentiable at $x = a$. The approximate absolute change in the value of f when x changes from a to $a + dx$ is

$$df = f'(a)\, dx$$

Estimating Relative Change

Differential Estimate of Relative Change

Let $f(x)$ be differentiable at $x = a$. The approximate relative change in the value of f when x changes from a to $a + dx$ is

$$\frac{df}{f(a)} = \frac{f'(a)}{f(a)}\,dx$$

Unclogging Arteries

In the late 1830s, the French physiologist Jean Poiseuille discovered that when blood flows along a blood vessel, the flux F (the volume of blood per unit time that flows past a given point) is proportional to the fourth power of the radius R of the blood vessel:

$$F = kR^4$$

How will a 10% increase in R affect F?

Sensitivity to Input

The equation

$$df = f'(x)dx$$

tells how sensitive the output of f is to a change in input at different values of x. The larger the value of f' at x, the greater the effect of a given change dx.

Finding the Depth of a Well

You want to calculate the depth of a well from the equation $s = 16t^2$ by timing how long it takes a heavy stone to splash into the water below. How sensitive will your calculations be to a 0.1 sec error in measuring the time?

Best Linear Approximation

Why do we use linearization?

The linearization of a function f at $x = a$ is the *best linear approximation* of f near $x = a$ in the sense that, if $g(x) = m(x - a) + c$ were any other linear approximation of f at $x = a$ with $g(a) = f(a)$ and negligible error compared with $x - a$, then $g(x) = f'(a)(x - a) + f(a)$.