A Modular Presentation System for the Calculus Sequence

2.5 Continuity

Yaw Chang

Michael Freeze

Mathematics and Statistics UNC-Wilmington

Continuity at a Point

Definition

A function f is **continuous at a number** a if

$$\lim_{x \to a} f(x) = f(a)$$

This definition hides three conditions in its statement. For f to be continuous at a, all three of the following must hold:

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

Identifying Continuous Functions

QUESTION: What classes of functions are continuous?

Remark: A function f is *continuous on an interval* (a, b) if it is continuous at every number in the interval.

If f and g are continuous at $x = a$ and c is a constant, then the following functions are also continuous at $x = a$: $f \pm g$, cf, fg, and f/g if $g(a) \neq 0$.

Types of Discontinuity

There are several basic types of discontinuities:

1. Oscillating Discontinuity $\sin\left(\frac{\pi}{x}\right)$ at $x = 0$

2. Jump Discontinuity $\frac{|x-1|}{x-1}$ at $x = 1$

3. Infinite Discontinuity $\frac{1}{x-2}$ at $x = 2$

4. Removable Discontinuity $\frac{x^2-9}{x-3}$ at $x = 3$

Composites

Moving a Limit Through a Function

If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

Example

Find $\lim_{x\to 1} \arcsin(\frac{1-\sqrt{x}}{1-x})$.

Solution: Because $\arcsin(x)$ is a continuous function, we can apply the above theorem:

$$\begin{aligned}
\lim_{x\to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right) &= \arcsin\left(\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}\right) \\
&= \arcsin\left(\lim_{x\to 1} \frac{(1-\sqrt{x})(1+\sqrt{x})}{(1-x)(1+\sqrt{x})}\right) \\
&= \arcsin\left(\lim_{x\to 1} \frac{1}{1+\sqrt{x}}\right) \\
&= \arcsin\frac{1}{2} = \frac{\pi}{6}
\end{aligned}$$

The Intermediate Value Theorem

The Intermediate Value Theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

Example

Show that there is a solution of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2.

Proof

1. $f(x) = 4x^3 - 6x^2 + 3x - 2$ is a continuous function on $[1, 2]$. (Why?)
2. $f(1) = -1$ and $f(2) = 12$.
3. Because $f(1) < 0 < f(2)$, by Intermediate Value Theorem, there is a number c between 1 and 2 such that $f(c) = 0$. i.e. the above equation has a solution between 1 and 2.

Application

QUESTION: Is any real number exactly 2 less than its cube?