A Modular Presentation System for the Calculus Sequence

2.4 The Precise Definition of a Limit

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Formal Definition of Limit

Definition

Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the **limit of $f(x)$ as x approaches a is L**, and we write

$$\lim_{x \to a} f(x) = L$$

if for every given number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$$

Finding Deltas Graphically

EXAMPLE: Use a graph to find a number δ such that

$$|(5x-3)-2| < 0.5 \quad \text{whenever} \quad |x-1| < \delta$$

Let's see this maple demo.

EXAMPLE: Use a graph to find a number δ such that

$$|(3x^2-x)-10| < 0.84 \quad \text{whenever} \quad |x-2| < \delta$$

Finding Deltas Algebraically

EXAMPLE: Show that $\lim_{x\to 1}(5x-3)=2$.

EXAMPLE: Show that $\lim_{x\to 2}(3x^2-x)=10$.

One-side Limit

Definition

- **Left-Hand Limit:**
  $\lim_{x \to a^-} f(x) = L$ if and only if for every $\varepsilon > 0$, there is a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad a - \delta < x < a$$

- **Right-Hand Limit:**
  $\lim_{x \to a^+} f(x) = L$ if and only if for every $\varepsilon > 0$, there is a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad a < x < a + \delta$$

Infinite Limits

Definition

$$\lim_{x \to a} f(x) = \infty$$

if and only if for every positive number M, there is a corresponding positive number δ such that

$$f(x) > M \quad \text{whenever} \quad |x - a| < \delta$$

An Example

Consider the function $f(x) = 1/x^2$. Clearly, $\lim_{x\to 0} f(x) = \infty$. The following table illustrates the relation about M and δ.

M	δ	Result
100	0.1	when $\|x-0\| < 0.1$, $f(x) > 10$
10000	0.01	when $\|x-0\| < 0.001$, $f(x) > 10000$
1000000	0.001	when $\|x-0\| < 0.001$, $f(x) > 1000000$
...	...	...