A Modular Presentation System for the Calculus Sequence

2.2 The Limit of a Function

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Informal Definition of Limit

Definition

We write

$$\lim_{x \to a} f(x) = L$$

and say

"the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x sufficiently close to a (on either side of a) but not equal to a.

Examples

EXAMPLE: Guess the value of $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2}$.

EXAMPLE: Guess the value of $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$.

One-Sided Limits

Definition

We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-hand limit of** $f(x)$ **as** x **approaches a** [or the **limit of** $f(x)$ **as** x **approaches** a **from the left**] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a.

Examples

EXAMPLE: Find $\displaystyle\lim_{x\to 2^-} \frac{|x-2|}{x-2}$.

EXAMPLE: Find $\displaystyle\lim_{x\to 0^+} \sin\left(\frac{\pi}{x}\right)$.

Splitting Limits by Sides

$$\lim_{x\to a} f(x) = L$$

if and only if

$$\lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L$$

Example

EXAMPLE: Sketch the graph of the following function and use it to determine the values of a for which $\lim_{x \to a} f(x)$ exists:

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ 2x+1 & \text{if } 0 < x \leq 1 \\ 3x & \text{if } x > 1 \end{cases}$$

Infinite Limits

Definition

Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

Examples

EXAMPLE: Find $\lim\limits_{x\to 2^-} \dfrac{3x-5}{x-2}$.

EXAMPLE: Find $\lim\limits_{x\to 2^+} \dfrac{3x-5}{x-2}$

Vertical Asymptotes

Definition

The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if

$$\lim_{x \to a^{\pm}} f(x) = \pm\infty$$

Examples

EXAMPLE: Find the vertical asymptote(s) of

$$f(x) = \frac{2x^2 - 5x + 7}{x^2 - 9}$$

EXAMPLE: Find the vertical asymptote(s) of

$$f(x) = \sec x$$