Maximum Flow in Networks

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Reference clrs, Chapter 26, Page 643-
Robert Sedgewick, Page 485
Udi Manber Page -

Introduction to Problem: Max Flow

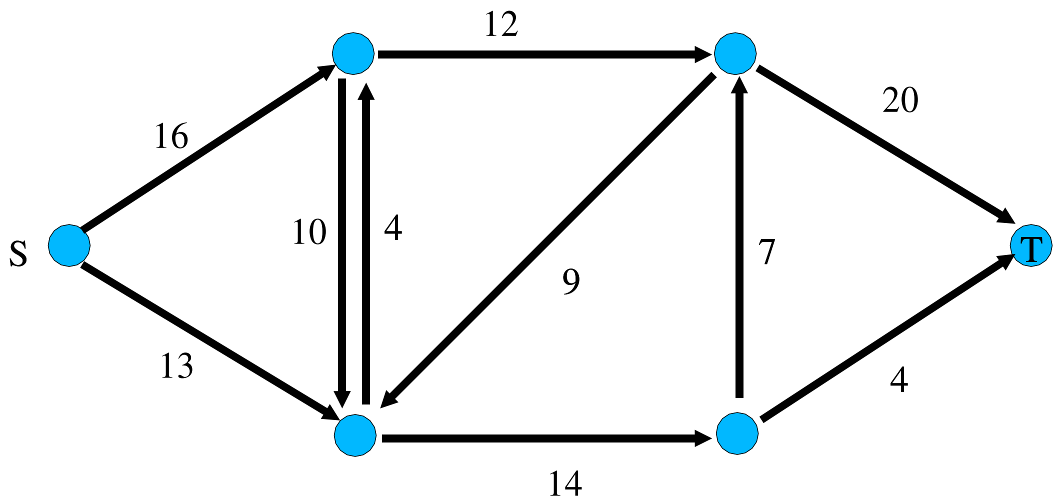
Given a tuple $G = (V, c, s, t)$, where V is the set of vertices, $s, t \in V$ are distinguished vertices called the *source* and the *sink* respectively, and c is a function $c : V^2 \to R^+$ assigning a nonnegative real capacity to each pair of vertices. We make G into a directed graph defining the set of directed edges

$$E = \{(u, v) \mid c(u, v) > 0\}$$

Edges can be thought of as wires or pipes along which electric current or fluid can flow. The capacity represents the carrying capacity of wires or pipes, say in amps or gallons per minute.

Problem Statement: Max Flow

The *max flow problem* is to determine the maximum possible flow that can be pushed from s to t, and to find a routing that achieves this maximum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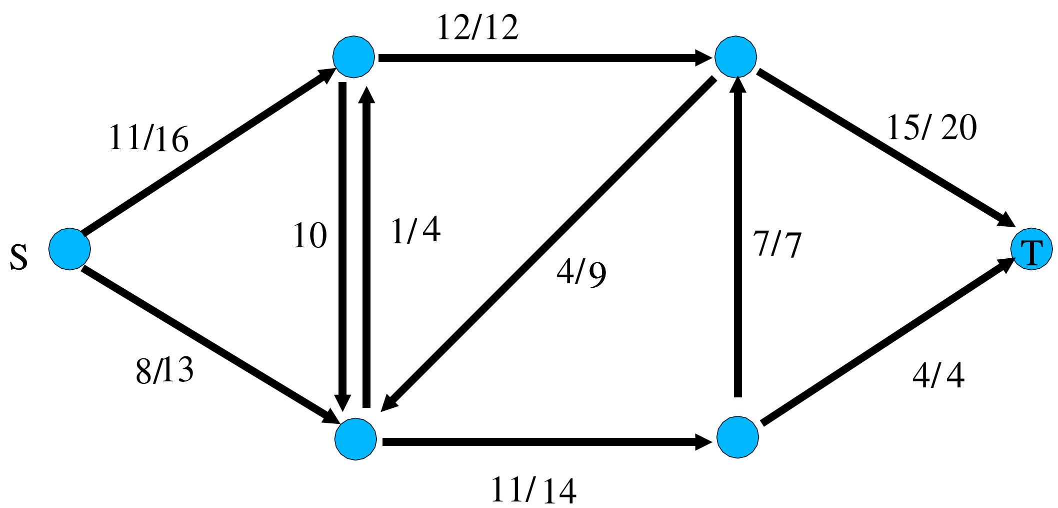
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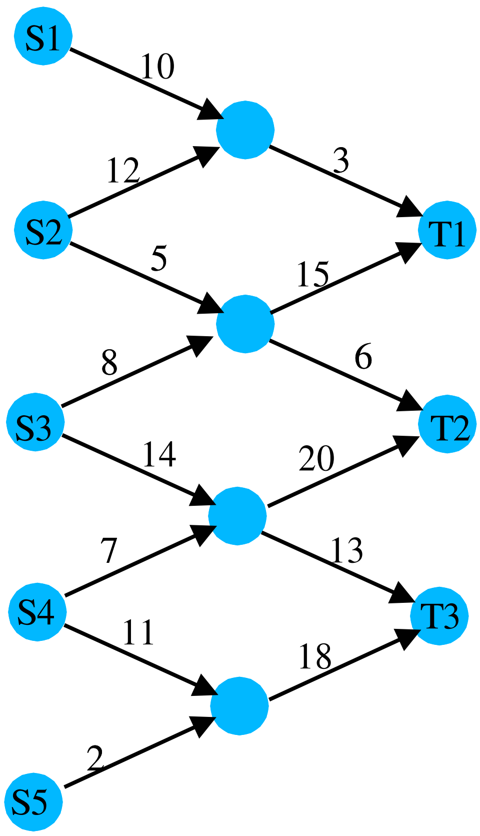
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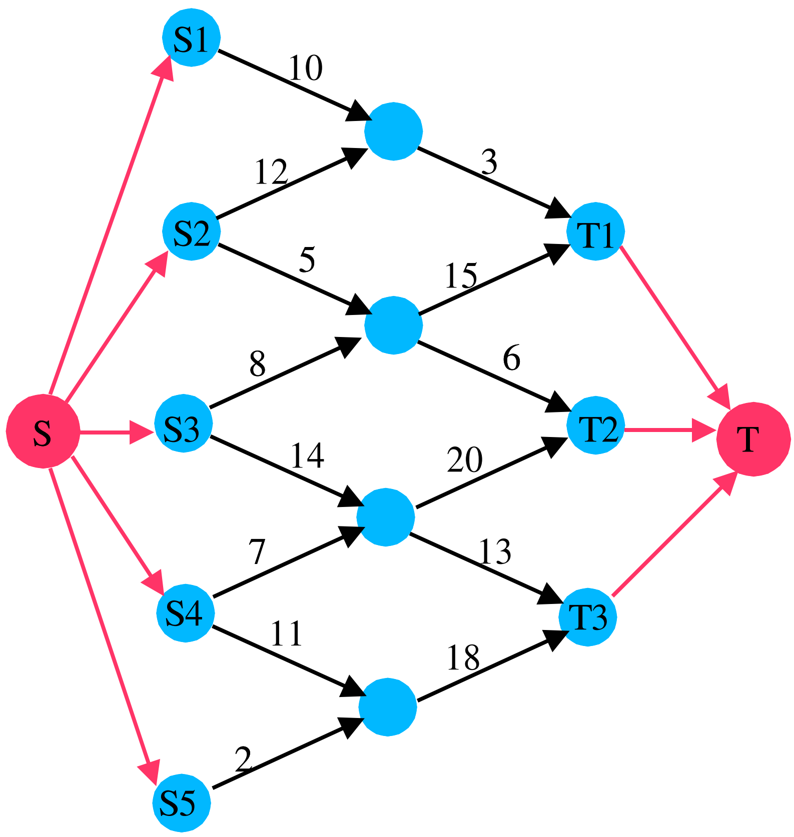
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Definitions: flow

A function $f : V^2 \rightarrow R$ is called a *flow* if the following three conditions are satisfied.
(a). *skew symmetry*: for all $u, v \in V$

$$f(u, v) = -f(v, u)$$

(b). *conservation of flow at interior vertices*: for all vertices u not in $\{s, t\}$

$$\sum_{v \in V} f(u, v) = 0$$

That is, *net flow* (total flow out minus total flow in) at any interior vertex u is 0.
(c). *capacity constraint*: for all u, v

$$f(u, v) \leq c(u, v)$$

Definitions: cut

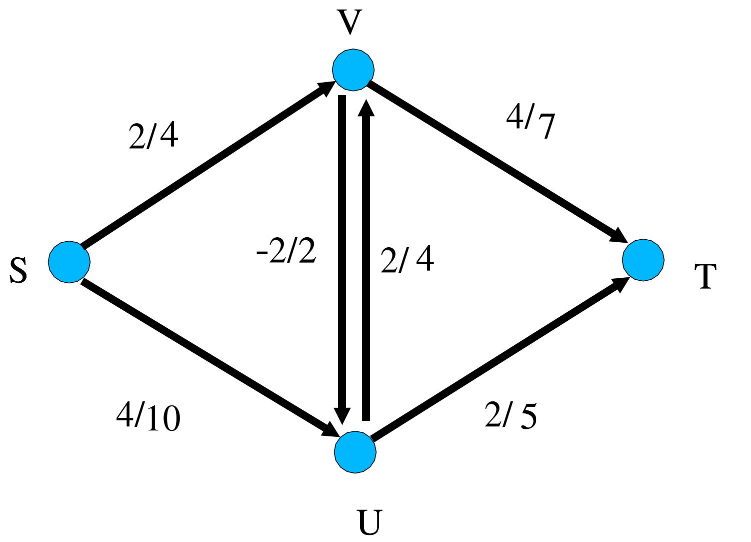
An $s, t - cut$ (or just cut when s, t are understood) is a pair A, B of disjoint subsets of V whose union is V and $s \in A$, $t \in B$. The *capacity of the cut* denoted $c(A, B)$, is:

$$c(A, B) = \sum_{u \in A, v \in B} c(u, v)$$

flow across the cut is similarly defined as:

$$f(A, B) = \sum_{u \in A, v \in B} f(u, v)$$

$f(A, B)$ is the sum of the positive flow values on edges from A to B minus the sum of the positive flow values on edges from B to A.


V
2/4
4/7
S
-2/2
2/ 4
T
4/10
2/ 5
U

Definitions: Value of flow in a network

The *value* of a flow f, in a network denoted $\mid f \mid$, is:

$$\mid f \mid = f(\{s\}, V - \{s\}) = f(V - \{t\}, \{t\})$$

or in other words the net flow out of s is same as net flow into the sink.
In the illustration the flow is 6.

Note: Although the definition gives the value of flow with respect to the cut $\{s\}$, $V - \{s\}$, (and $V - \{t\}, \{t\}$) the flow is the same no matter where it is measured.

Definitions: flow across a cut

For <u>any</u> s, t-cut A, B and flow f

$$| f | = f(A, B)$$

$$| f | \leq c(A, B)$$

Definitions: Residual Capacity of an edge

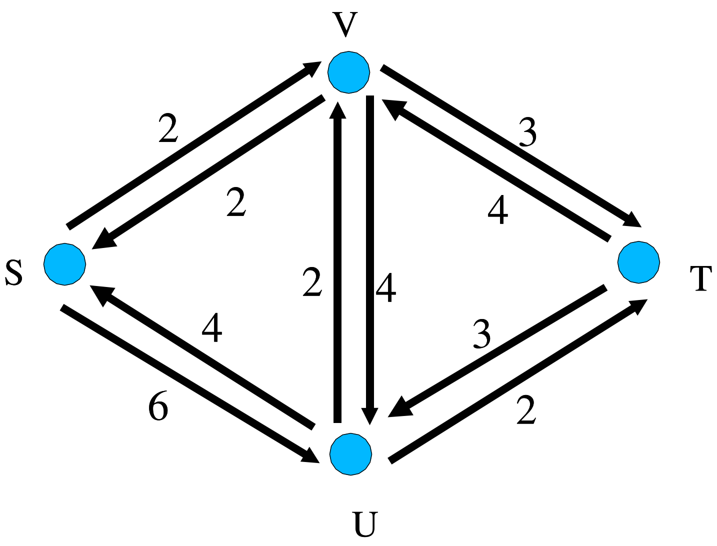
Given a flow f on G with capacities c, the residual capacity r is defined for each pair of vertices as:

$$r = c - f$$

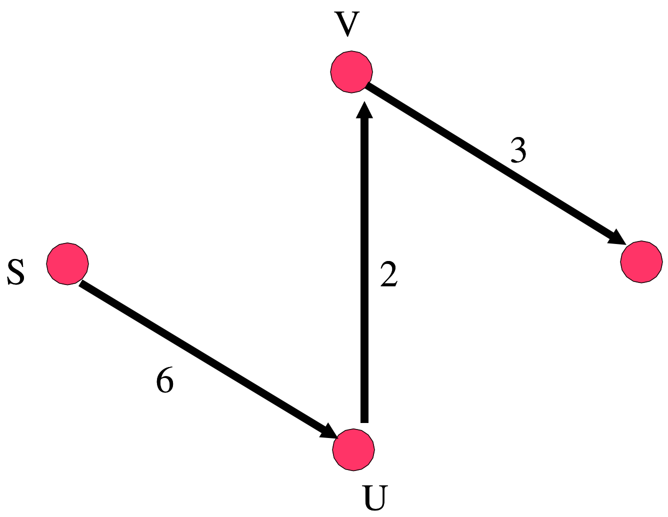
The *residual graph* associated with $G = (V, E, c)$ and flow f is the graph $G_f = (V, E_f, r)$, where

$$E_f = \{(u, v) \mid r(u, v) > 0\}$$

The residual capacity $r(u, v)$ represents the amount of additional flow that could be pushed along the edge (u, v) without violating the capacity constraint.
Note: When the flow is negative, the residual capacity can be greater than the capacity. For example, if $c(u, v) = 16$ and $f(u, v) = -4$ the residual capacity is $16 - (-4) = 20$



Residual Graph



Augmenting Path

Definitions: Augmenting Path

Given G and flow f on G, an *augmenting path* is a directed path from s to t in the residual graph G_f.

Note: An augmenting path represents a sequence of edges on which the capacity exceeds the flow.

Definitions: Residual Capacity of an Augmenting Path

The maximum amount by which we can increase the flow on each edge in an augmenting path p is the *residual capacity* of p, given by:

$$c_f(p) = \text{min}\{c_f(u, v) : (u, v) \text{ is on } p\}$$

FORD-FULKERSON-METHOD(G, s, t)

```
initialize flow f to 0
while there exists an augmenting path p
    do augment flow f along p
return f
```

Reference clrs 651

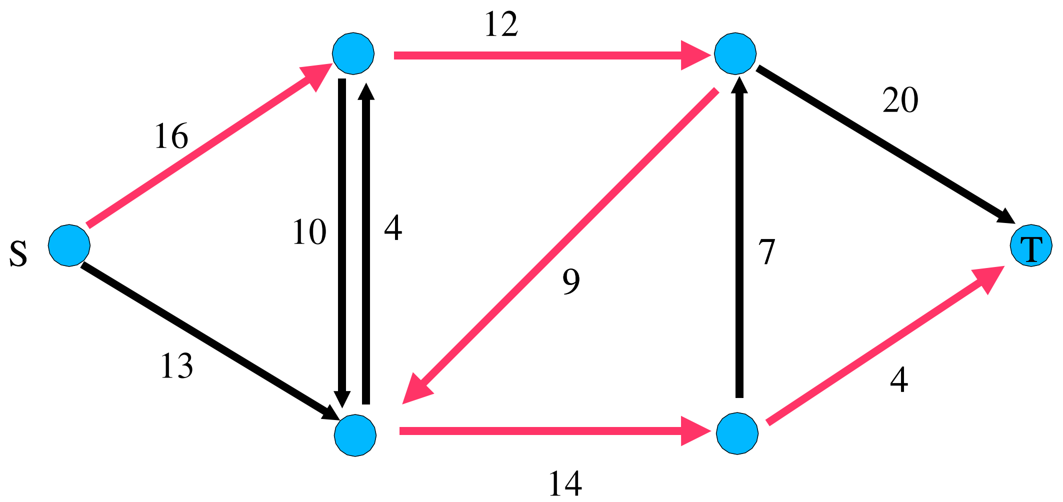
Max-Flow Minimum-Cut Theorem

If f is a flow network $G = (V, E)$ with source s and sink t, then the following conditions are equivalent:

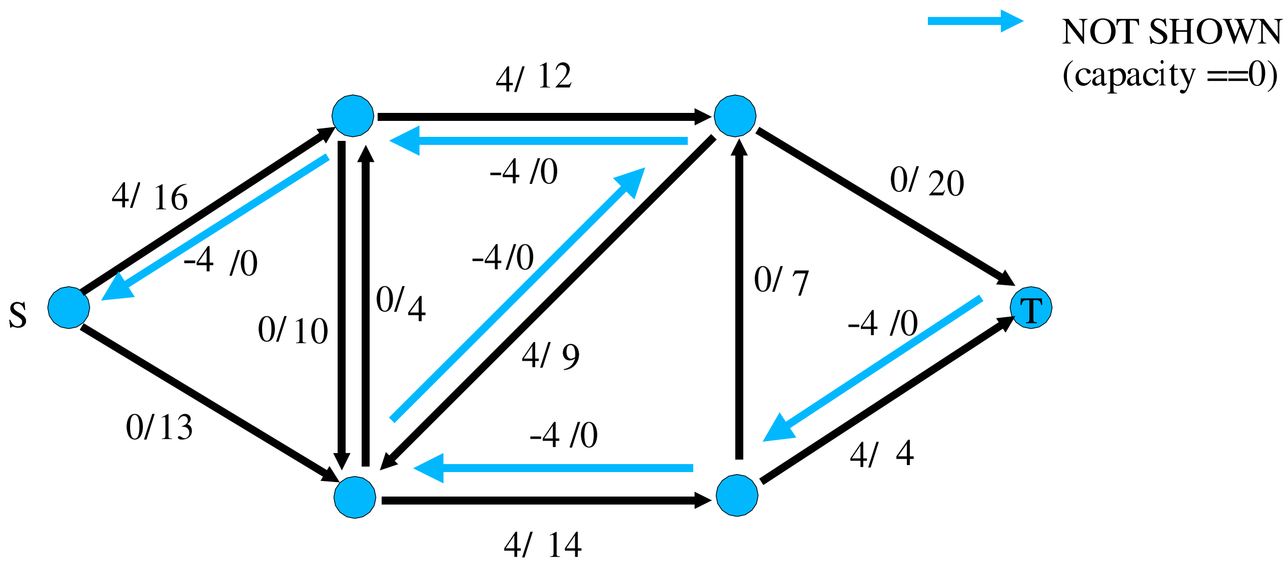
1. f is a maximum flow in G
2. The residual network G_f contains no augmenting paths
3. $\mid f \mid = c(S, T)$ for some cut (S, T) of G.

`Reference: clrs2e p-657`

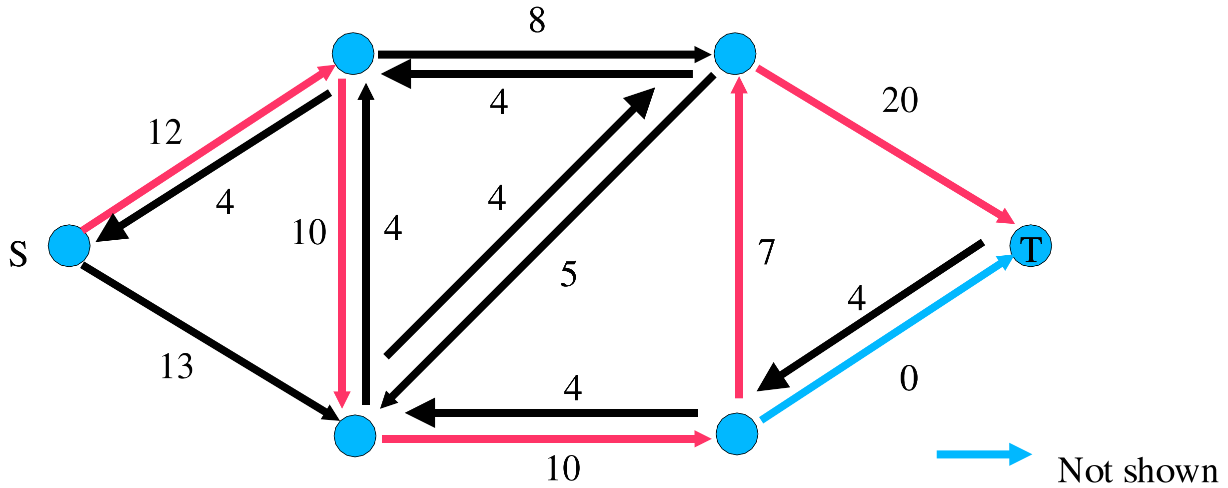
Network showing residual capacity



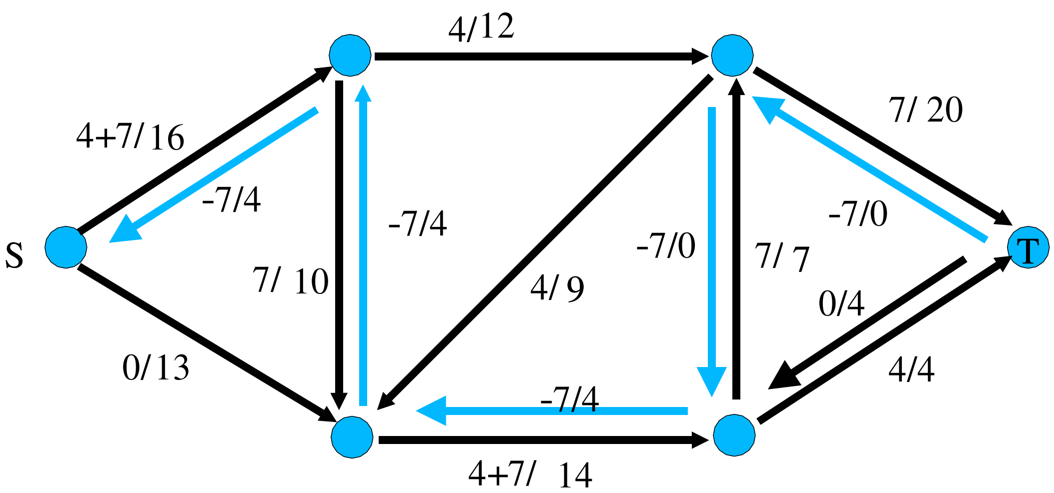
Network showing flows



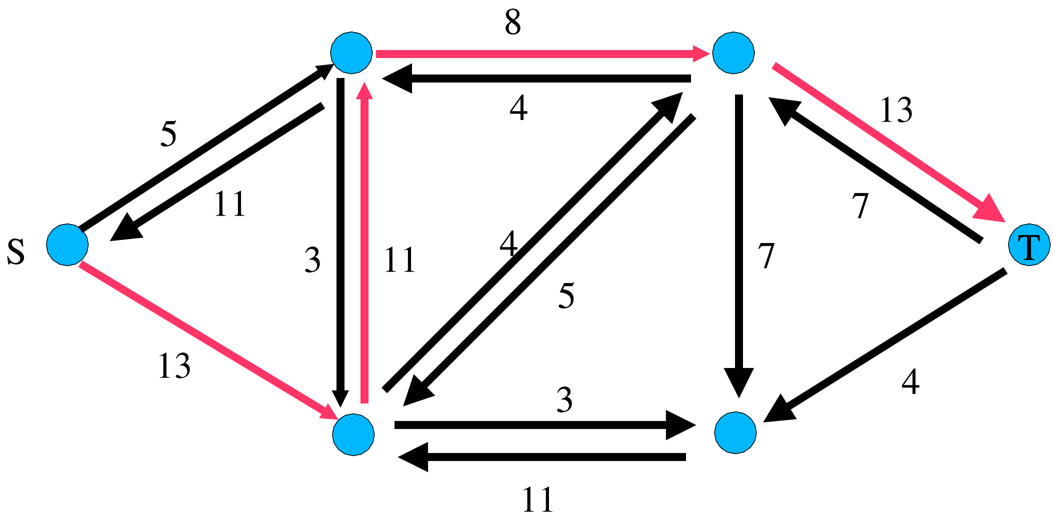
Network showing residual capacity



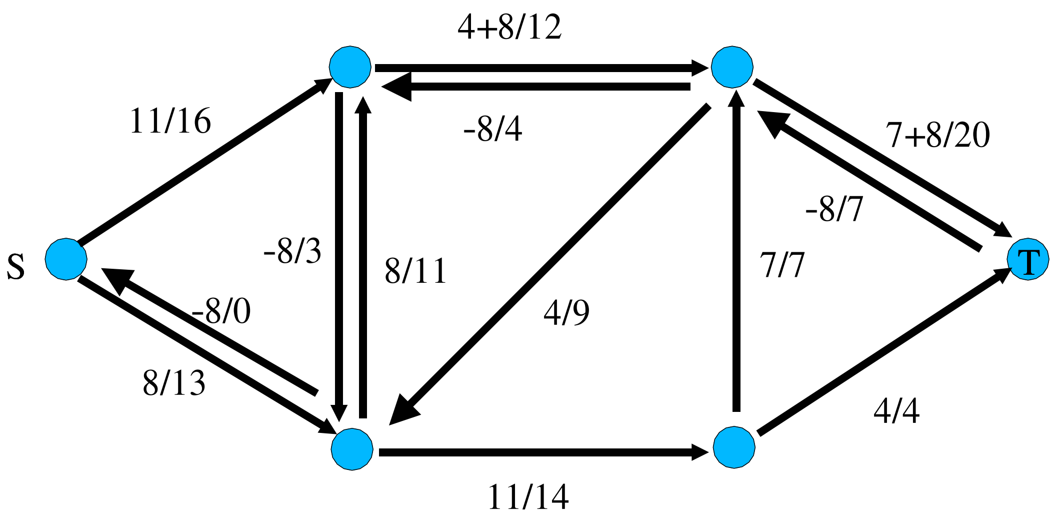
Network showing flows



Network showing residual capacity

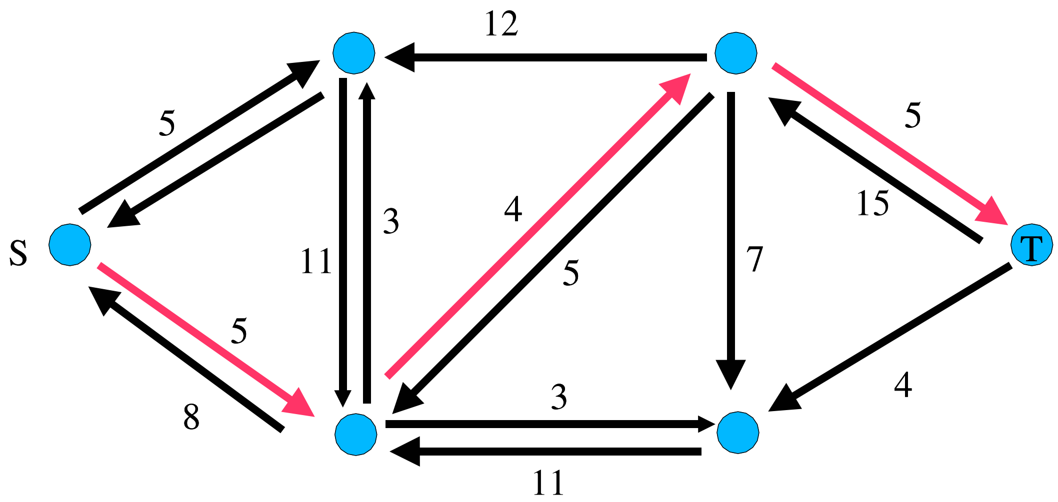


Network showing flows

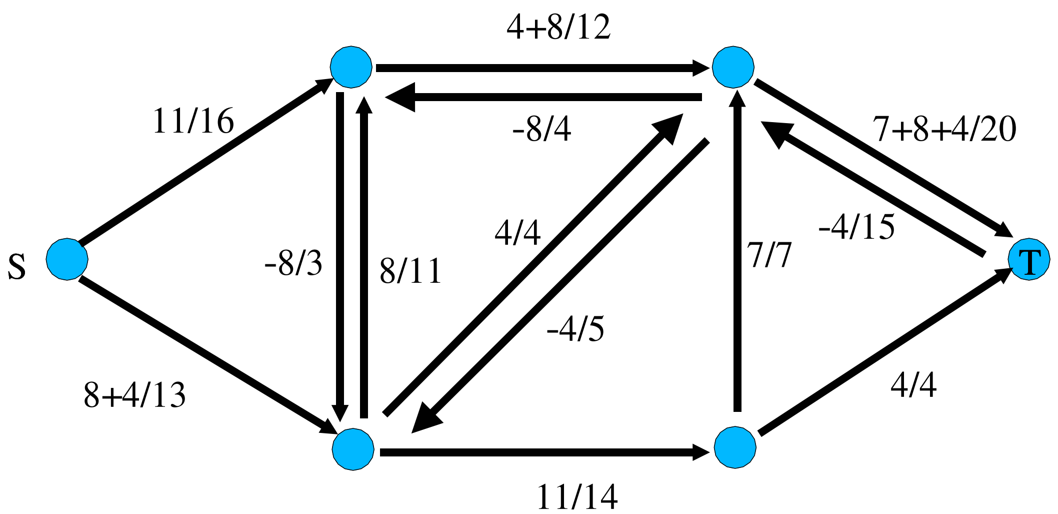


*note the difference from the book

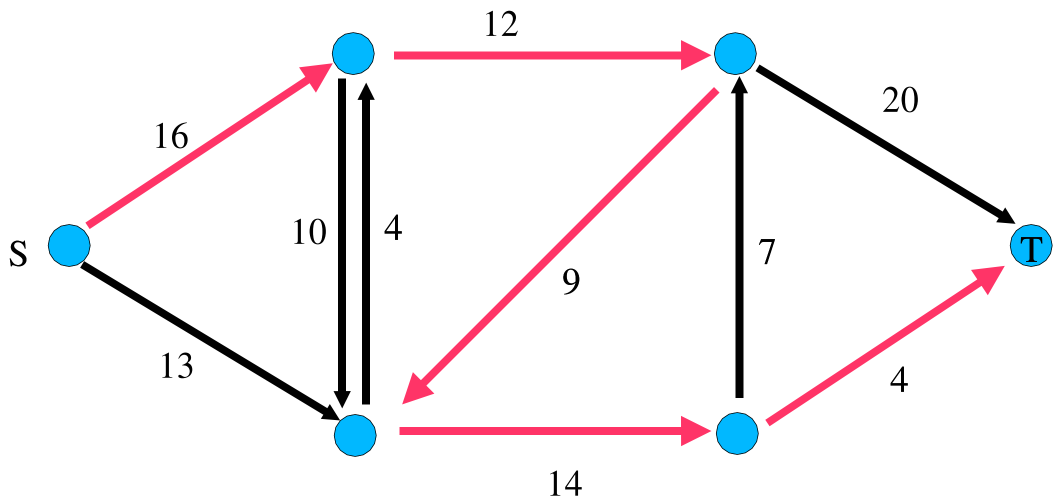
Network showing residual capacity



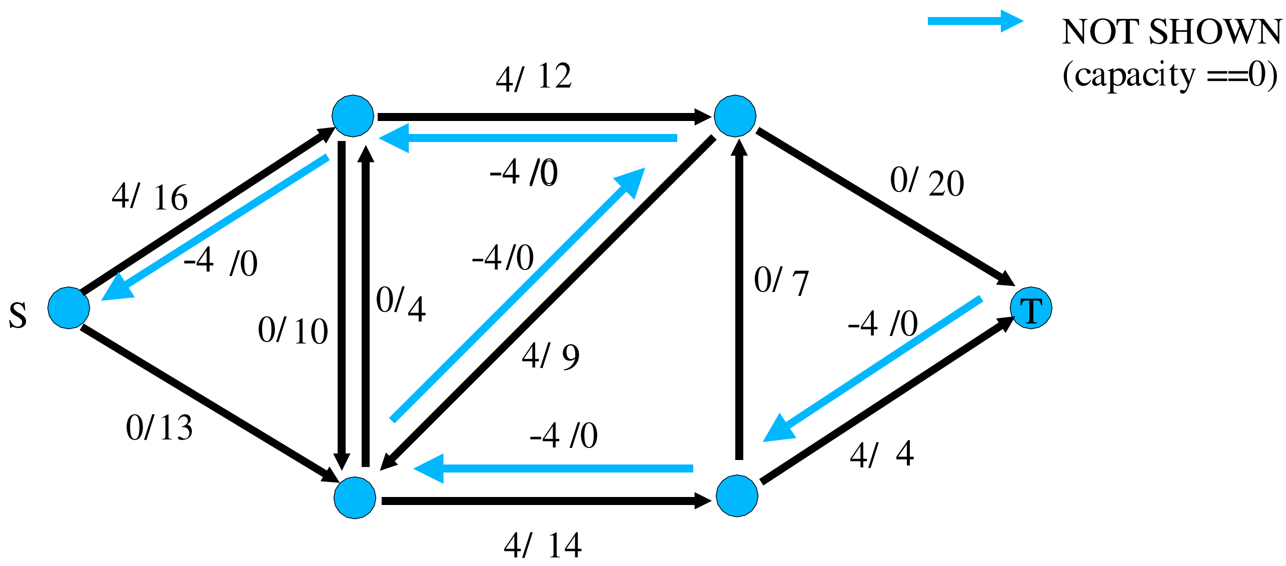
Network showing flows



Network showing residual capacity



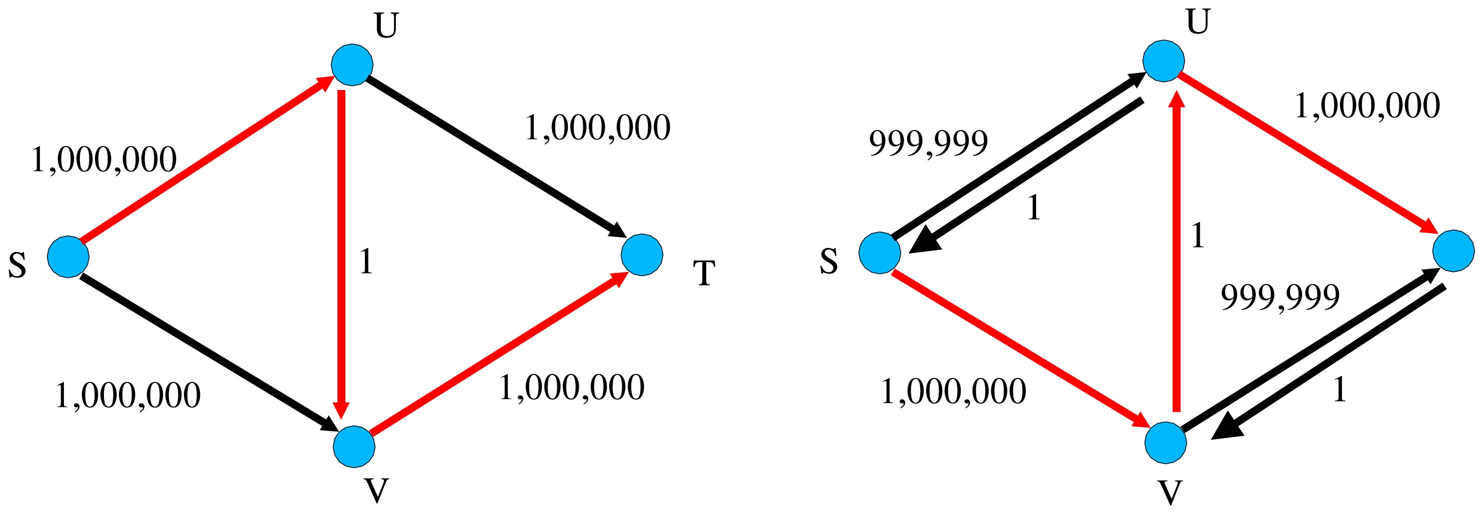
Network showing flows



FORD-FULKERSON-METHOD(G, s, t)

for each edge $(u, v) \in E(G)$
    do $f[u, v] \leftarrow 0$
        $f[v, u] \leftarrow 0$
    while there exists a path p from s to t in the residual
        do $c_f(p) \leftarrow$ min$\{c_f(u, v)$: (u, v) is in $p\}$
            for each edge (u, v) in p
                do $f[u, v] \leftarrow f[u, v] + c_f(p)$
                      $f[v, u] \leftarrow -f[u, v]$
                      $c_f(u, v) \leftarrow c_f(u, v) - f[u, v]$/* Adhar */
                      $c_f(v, u) \leftarrow c_f(v, u) - f[v, u]$/* Adhar */

`Reference clrs 651`



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Edmond Karp Algorithm

Worst case complexity of FORD FULKERSON algorithm is $O(E \mid f^* \mid)$.

The complexity of FORD FULKERSON algorithm is improved by implementing the computation of the augmenting path (in line 4) with a breadth first search. That is, the augmenting path is the shortest path from s to t in the residual network.
The resulting algorithm is called Edmond Karp Algorithm

`Reference clrs 660`

Network Flow Problem : Revisited

Let $\bar{x} = x_1, x_2, \ldots, x_n$ represent the values of the flow for all the edges (n is the number of edges here). The objective function is the value of the total flow in the network:

$$c(\bar{x}) = \sum_{i \in S} x_i$$

where S is the set of edges leaving the source. subject to the constraints:

- The flow x_i in a link is limited by its capacity c_i.

  $$x_i \leq c_i$$

  for all i, $1 \leq i \leq n$
- The conservation constraint- the flow into a node and out of the node are equal.

  $$\sum_{x_i outofv} x_i - \sum_{x_j intov} x_j = 0$$

  for all $v \in V - \{s, t\}$