Matching

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Reference clrs, Chapter 26, Page 664-

Definitions: Matching

Given an undirected, connected graph,
$G = (V, E)$,
a *matching* is a set of edges no two of which have a vertex in common.

A vertex that is not incident to any edge in the matching is called *unmatched*.

A *perfect matching* is one in which all vertices are matched.
A *maximum matching* is one with the maximum number of edges.
A *maximal matching* is a matching that cannot be extended by the addition of an edge.
The reason for the name is that an edge can be thought of as a match of two vertices. We insist that no vertex belongs to more than one edge from the matching so that it is a monogamous matching.

Problem involving matching occurs in many situations (besides social). Medical students to hospitals, Workers may be matched to jobs, machine to parts, courses to rooms, and so on.

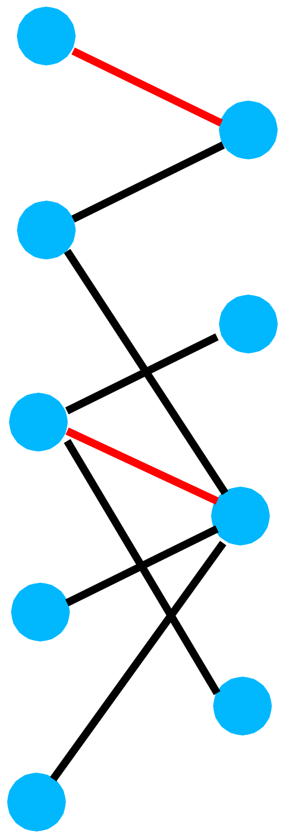
Problem Statement: Maximum Bipartite Matching

Given an undirected graph $G = (V, E)$ a **matching** M is a subset of edges $M \subseteq E$ such that for vertices $v \in V$, at most one edge of M is incident on v.

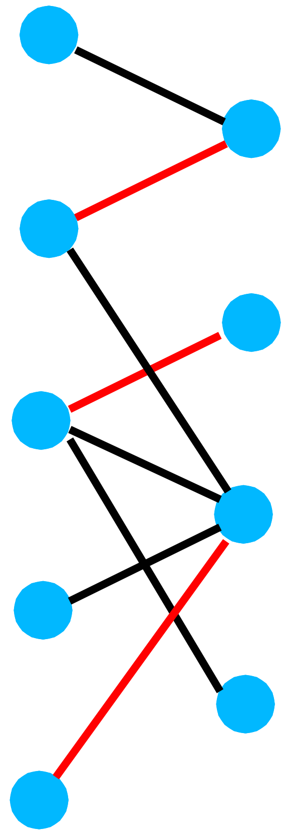
A **maximum matching** is a matching of maximum cardinality. That is, a matching M such that for any other matching $M^{'}$ it is true that $\mid M \mid \geq \mid M^{'} \mid$.

`Note: We restrict our attention to finding matching in bi-partite graphs.`

`Reference clrs 664`



Matching with Cardinality 2



Matching with Cardinality 3

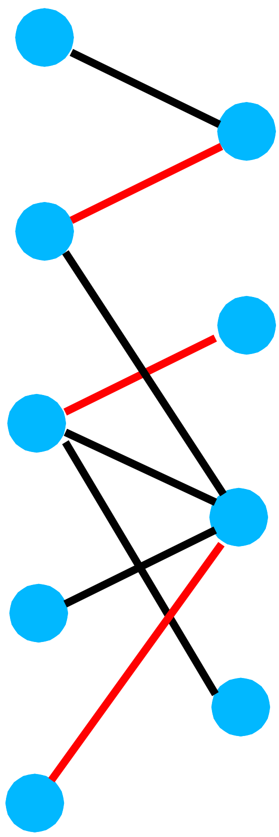
Maximum Bipartite Matching: Construction of a Flow Network

$$V^{'} = V \cup \{s,t\}$$

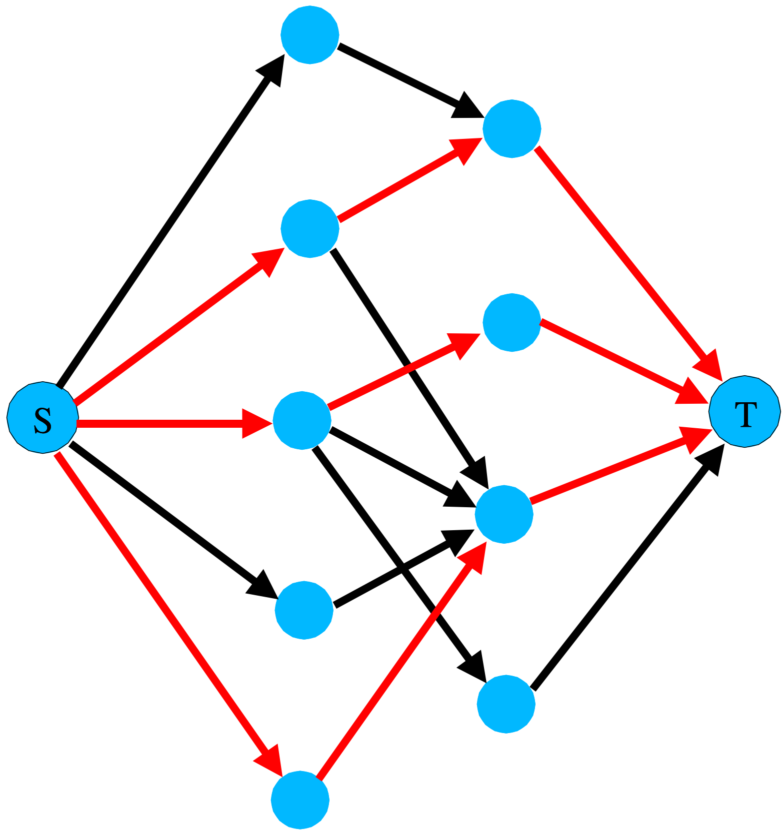
$$E = \begin{array}{l} \{(s,u) : u \in L\} \\ \cup\{(v,t) : v \in R\} \\ \cup\{(u,v) : u \in L, v \in R, (u,v) \in E\} \end{array}$$

Capacity of each edge in $E^{'}$ is assigned unity.

`Reference clrs 664`



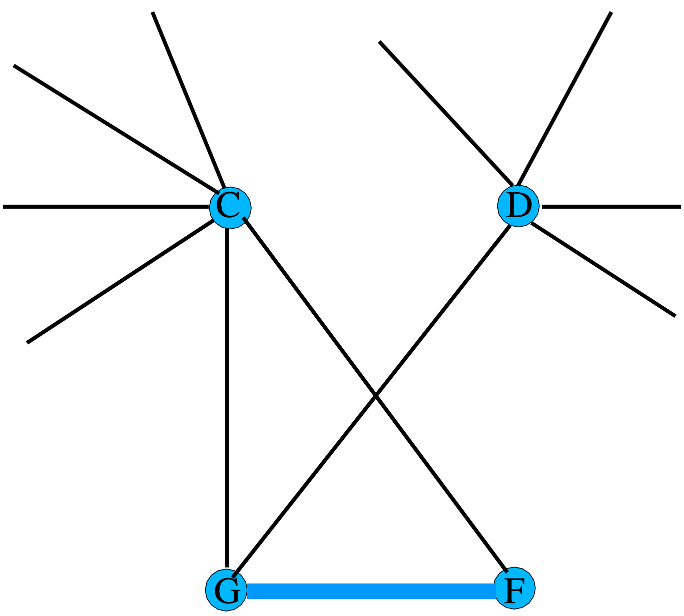
Maximum Matching



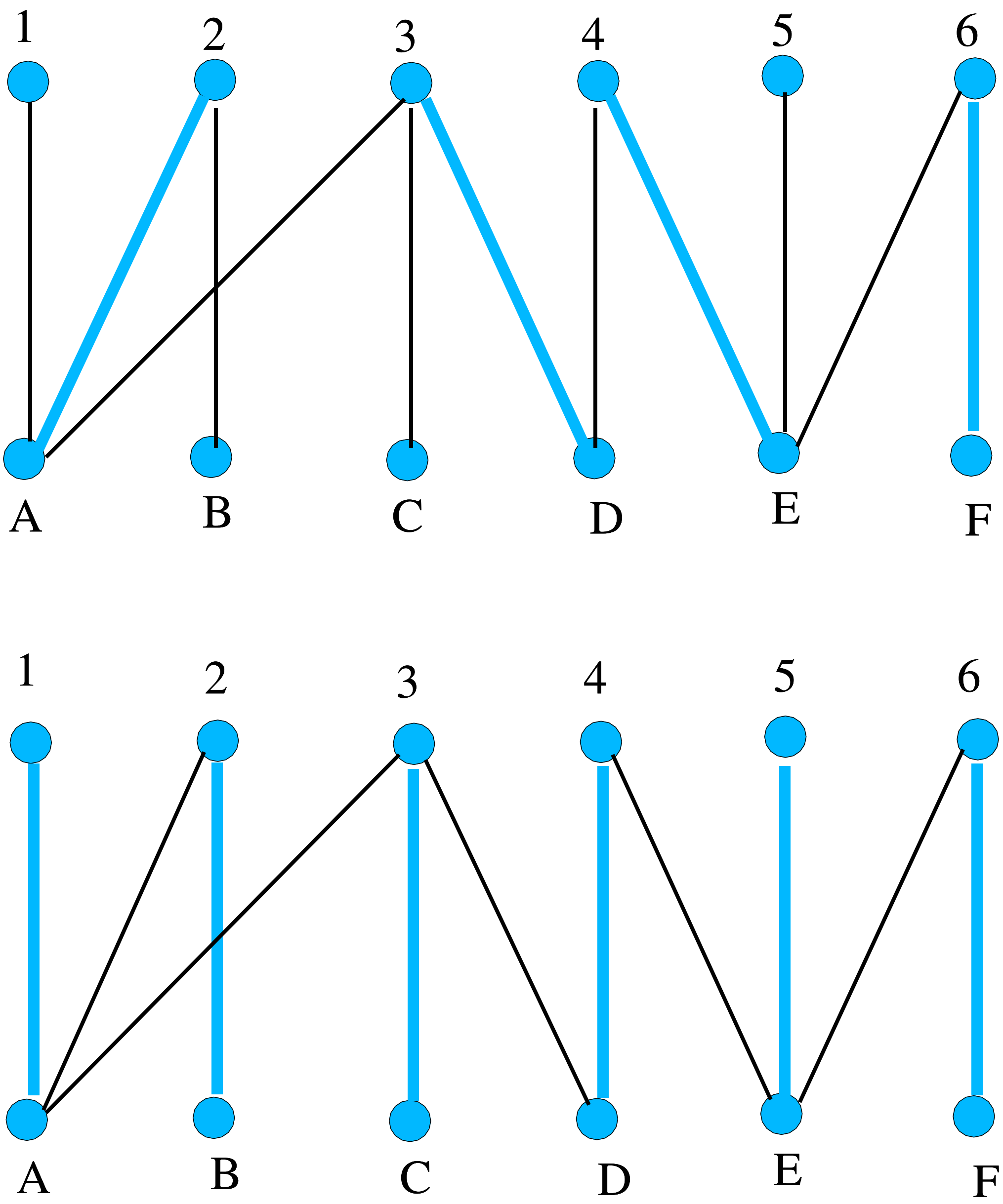
Flow Network with Maximum Flow

BiPartite Matching

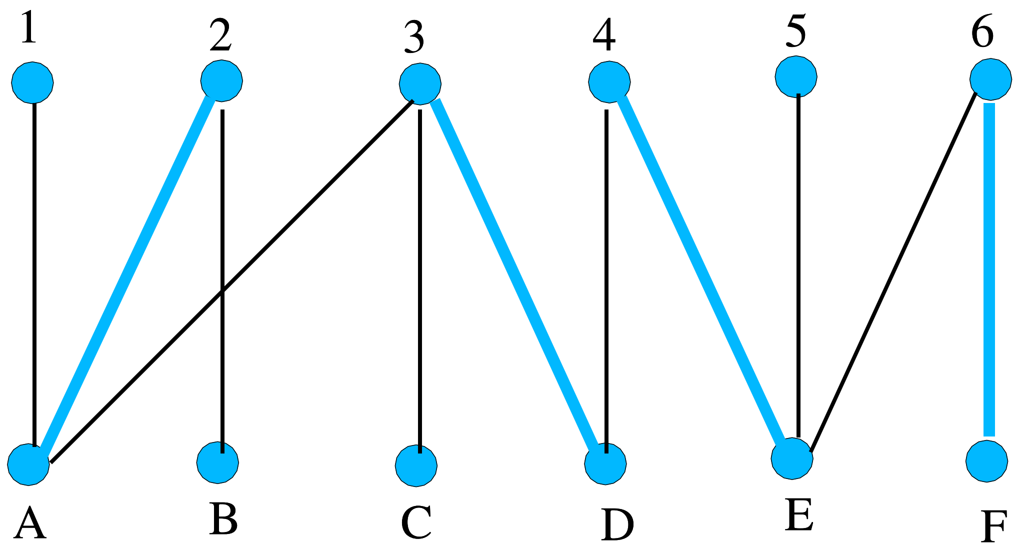
Consider a undirected bipartite graph $G = (V, E, U)$, such that V and U are disjoint set of vertices, and E is a set of edges connecting vertices in V with vertices in U.
Problem: Find a maximum-cardinality matching in a bipartite graph.

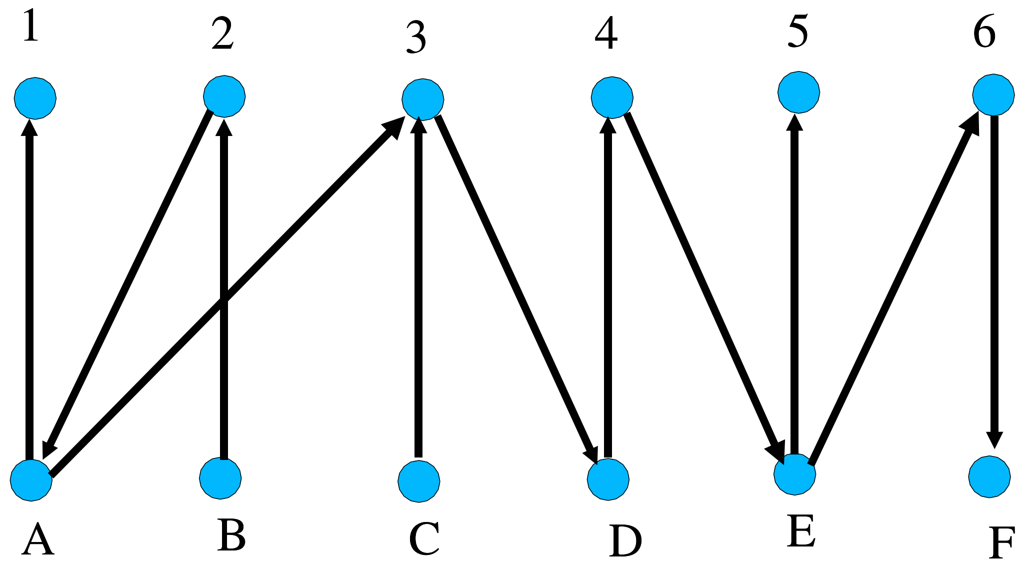


Extending a matching



Extending a bipartite matching





Finding Alternating Paths

Philanthropist Problem : Revisited

There are n organizations that want to contribute money to k computer science departments!!! The goal is to maximize the contributions which are subject to the constraints. This problem is a generalization of the matching problem which also has the following linear programming formulation.
There are nk variables x_{ij}, $1 \leq i \leq n$, $1 \leq j \leq k$, representing the amount of money organization i is willing to give to department j. The objective is to maximize:

$$\sum_{i,j} x_{ij}$$

subject to following constraints:

- Each organization i has a limit of s_i on its total contribution for the year.

$$\sum_{j=1}^{k} x_{ij} \leq s_i$$

- Each organization i also has a limit a_{ij} on what it is willing to contribute to the department j.

$$x_{ij} \leq a_{ij}$$

- Suppose that each department j also has a limit t_j on what it is willing to receive (somewhat unrealistic, but interesting nevertheless!)

$$\sum_{i=1}^{n} x_{ij} \leq t_j$$

Assignment Problem: Revisited

Change the philanthropist problem as follows.
Each organization can donate money *to only one department and, each department can accept money from only one organization*.
It is a *matching problem with weights*. Each possible match has a dollar amount attached to it, and we want to find not only maximum matching but also one that maximizes total donation. This problem is known as **weighted matching problem** or **assignment problem**.
There are nk variables x_{ij}, $1 \leq i \leq n$, $1 \leq j \leq k$, representing the amount of money organization i is willing to give to department j. The objective is to maximize:

$$\sum_{i,j} a_{ij} x_{ij}$$

subject to following constraints:

- Each organization can donate only to one department.

$$\sum_{i=1}^{n} x_{ij} \leq 1$$

- Each department can receive only from one organization.

$$\sum_{j=1}^{k} x_{ij} \leq 1$$

The variables can assume only integer values (either 0 or 1) (called integer linear programming).