0.1 Problem Set 2

1. Consider the system
\[ \begin{align*}
  x' &= -4x - y \\
  y' &= x - 2y.
\end{align*} \]

(a) Determine the second order differential equation satisfied by \( x(t) \).
(b) Solve the differential equation for \( x(t) \).
(c) Using this solution, find \( y(t) \).
(d) Verify your solutions for \( x(t) \) and \( y(t) \).
(e) Find a particular solution to the system given the initial conditions \( x(0) = 1 \) and \( y(0) = 0 \).

2. Consider the general system
\[ \begin{align*}
  x' &= ax + by \\
  y' &= cx + dy.
\end{align*} \]

Can one determine the family of trajectories for the general case? Recall, this means we have to solve the first order equation
\[ \frac{dy}{dx} = \frac{ax + by}{cx + dy}. \]

Actually, this equation is homogenous of degree 0. It can be written in the form \( \frac{dy}{dx} = F \left( \frac{y}{x} \right) \). For such equations, one can make the substitution \( z = \frac{y}{x} \) and obtain a separable equation for \( z \),
\[ x \frac{dz}{dx} = F(z) - z. \]

(a) Using the system, find the equation satisfied by \( z \).
(b) Solve for \( y = y(x) \) using your equation from the last part for the system
\[ \begin{align*}
  x' &= x - y \\
  y' &= x + y.
\end{align*} \]
(c) Describe the family of solutions obtained.
3. Consider the following systems. Determine the families of orbits for each system and sketch several orbits in the phase plane and classify them by their type (stable node, etc.)

(a)

\[ x' = 3x \\
   y' = -2y. \]

(b)

\[ x' = -y \\
   y' = -5x. \]

(c)

\[ x' = 2y \\
   y' = -3x. \]

(d)

\[ x' = x - y \\
   y' = y. \]

(e)

\[ x' = 2x + 3y \\
   y' = -3x + 2y. \]