Noncommutative harmonic analysis and mobile communications

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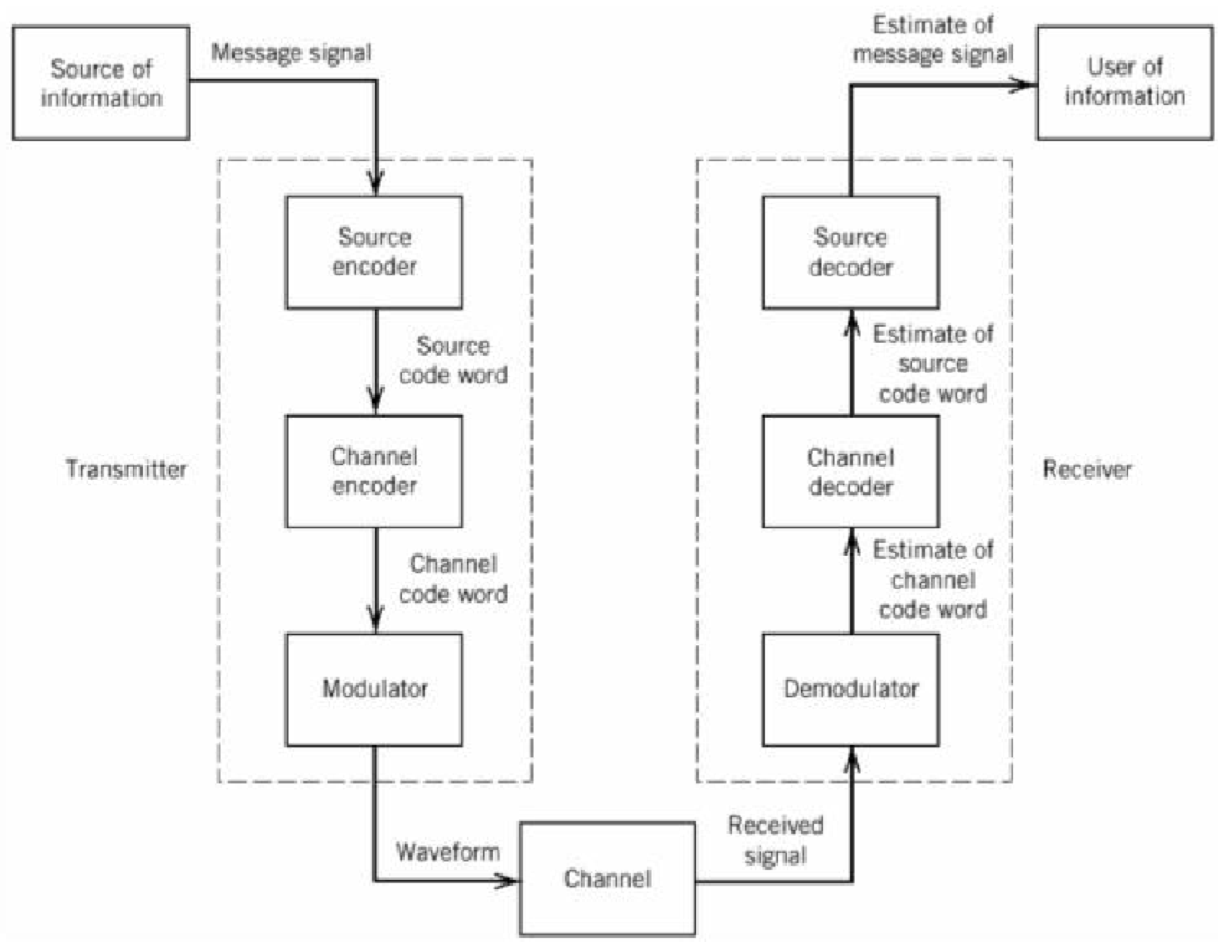
Agenda

- Wireless communications
- Noncommutative harmonic analysis
- Time-frequency analysis and pseudodifferential operators
- Approximate diagonalization, Banach algebras
- Mobile communications revisited
- Outlook

Basic communication scheme



Let $c = \{c_k\}_{k\in\mathcal{I}}$ be the discrete data to be transmitted. Let $\{\varphi_k\}_{k\in\mathcal{I}}$ be a family of bandlimited transmission pulses. The (continuous-time) signal to be transmitted is

$$x(t) = \sum_{k\in\mathcal{I}} c_k\varphi_k(t)$$

$\mathbf{H}$ is the operator representing the radio channel. The received signal is (in the absence of noise)

$$y = \mathbf{H}x$$

We extract the discrete data $d = \{d_l\}_{l\in\mathcal{I}}$ from y by computing

$$d_l = \langle y, \psi_l\rangle, \qquad l \in \mathcal{I}$$

where $\{\psi_k\}_{k\in\mathcal{I}}$ is a family of receiver functions. For simplicity we assume $\varphi_k = \psi_k$.

We introduce the coefficient operator C by

$$Cy = \{\langle y, \varphi_l \rangle\}_{l \in \mathcal{I}}$$

and note that $x = C^* c$. Then

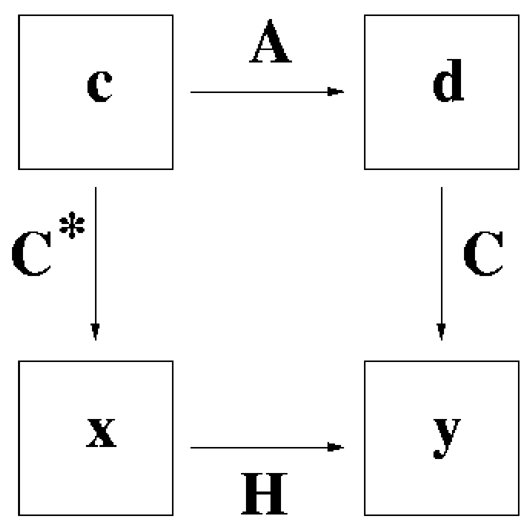
$$d = C\mathbf{H}C^* c$$

Defining the matrix $A = [A_{k,l}]_{k,l \in \mathcal{I}}$ where

$$A_{k,l} = \langle H\varphi_l, \varphi_k \rangle$$

and using linearity of $\mathbf{H}$ we get

$$d = C\mathbf{H}C^* c = Ac$$



Equalization...

... is the process of compensating for channel distortions at receiver.

We need to solve $Ac = d$.

Problems:

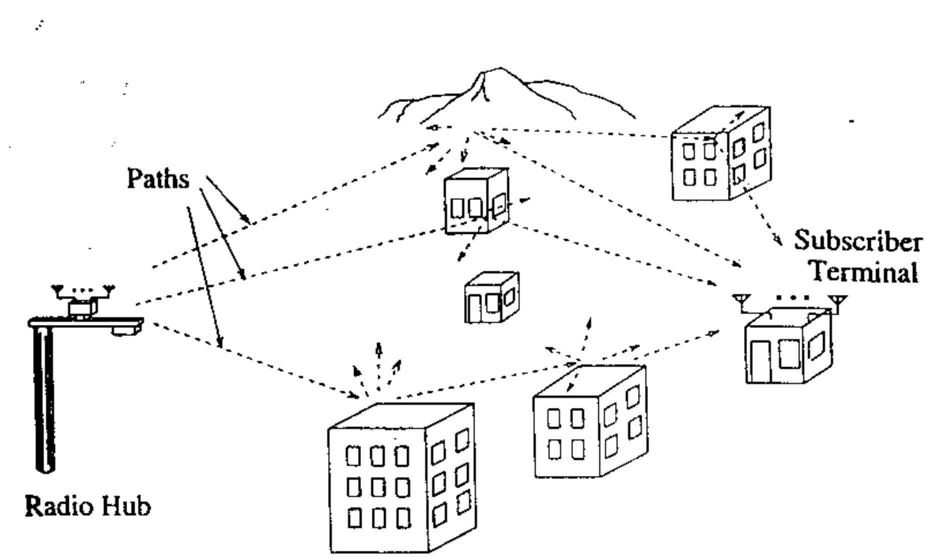
- A is an infinite-dimensional, or very large, matrix
- Application of A or inversion of A may be very expensive
- A may not be invertible
- The computation of C^*c and Cy may be expensive
- A and $\mathbf{H}$ are not known at the receiver (requires channel estimation)
- ...

Ideal case: choose φ_k's such that A is diagonal and computation of C^*c and Cy is cheap.

Fixed wireless communication

Transmitter and receiver are not moving.

Multipath propagation



Multipath propagation causes signal to arrive at receiver with different delays and different amplitudes: Transmission pulses are spread out in time ("delay spread").

Linear time-invariant channels

Let x denote the transmitted signal, the received signal y is given by (in absence of AWGN)

$$y(t) = (\mathbf{H}x)(t) = (h * x)(t) = \int_{-\infty}^{+\infty} h(t-s)x(s)ds$$

where $\mathbf{H}$ represents the linear operator modeling the wireless channel, h is the *impulse response*.

Clearly, we can express this equivalently as

$$y(t) = \int_{-\infty}^{+\infty} \widehat{h}(\omega)\widehat{x}(\omega)e^{2\pi i\omega t}\,d\omega$$

where $\widehat{h}(\omega)$ is the *transfer function*.

Emitted signal x is of the form

$$x(t) = \sum_k c_k \varphi_k,$$

where φ_k are transmission pulses (yet to be determined!).

We want a very simple equalizer at receiver.

$$d_l = \langle y, \varphi_l \rangle = \sum_k c_k \langle H\varphi_k, \varphi_l \rangle = \sum_k A_{l,k} c_k$$

If the φ_k are eigenvectors of H with eigenvalues λ_k and if the φ_k are mutually orthogonal, then

$$\langle H\varphi_k, \varphi_l \rangle = \langle \lambda_k \varphi_k, \varphi_l \rangle = \lambda_l,$$

thus A is diagonal with $A_{k,k} = \lambda_k$. Eigenvectors of H are $e^{2\pi i k \cdot}$, eigenvalues are $\lambda_k = \widehat{h}(k)$. Thus choose $\varphi_k(t) = e^{2\pi i k t}$, and we get the equalized data

$$c_l = \frac{d_l}{\widehat{h}(l)}$$

What are the key ingredients?

Time-invariance, convolution operators, **Fourier transform:** diagonalizes convolution operators, which makes everything much easier (capacity theorems, coding, equalization, numerical implementation,...)

We use Fourier transform in different roles:
(i) on $\mathbb{R}$ for modeling, physical realization
(ii) on $\mathbb{Z}, \mathbb{T}$ for A/D conversion/discretization
(iii) on $\mathbb{C}^N$ for actual implementation

We are dealing with harmonic analysis on locally compact *abelian* groups $\mathbb{G}$.

Mobile wireless communication

Moving transmitter and receiver cause channel $\mathbf{H}$ to be time-varying. Relative motion between transmitter and receiver results in Doppler effect. In addition to delay spread caused by multipath propagation, signals get spread out in frequency ("Doppler spread").

We can no longer diagonalize operator $\mathbf{H}$ by Fourier transform, Operators do not commute anymore. No transform will simultaneously diagonalize *all* mobile channels.

Goal: We want to design an orthonormal basis $\{\varphi_k\}_{k\in\mathcal{I}}$ for such that the matrix A with entries

$$A_{k,l} = \langle H\varphi_l, \varphi_k\rangle$$

is as diagonal as possible for a large class of mobile channels.

Idea: establish correspondence between mobile (=noncommutative) communications and "Fourier analysis" on noncommutative groups.

Integrated representation

What substitute for the Fourier transform does noncommutative harmonic analysis offer?

Let $\mathbb{H}$ be locally compact group, F a function in $\boldsymbol{L}^1(\mathbb{H})$ and $(\pi, \mathcal{H})$ a unitary representation of $\mathbb{H}$ (a homomorphism form $\mathbb{H}$ into the group of unitary operators on a Hilbert space $\mathcal{H}$). The *integrated representation* of F is

$$\pi(F) = \int_{\mathbb{H}} F(\mathbf{h})\pi(\mathbf{h})d\mathbf{h},$$

in the (weak) sense that for all $f, g \in \mathcal{H}$:

$$\langle \pi(F)f, g\rangle = \int_{\mathbb{H}} F(\mathbf{h})\langle \pi(\mathbf{h})f, g\rangle d\mathbf{h}.$$

Let $\mathbb{H}$ be the (polarized) Heisenberg group and $(\pi, \boldsymbol{L}^2(\mathbb{R}^d))$ be its Schrödinger representation. Then

$$\pi(F) = \int_{\mathbb{H}} F(\omega, x, \tau)e^{2\pi i\tau} M_\omega T_x d\omega dx d\tau$$

Mobile wireless channel can be written as

$$y(t) = \mathbf{H}x(t) = \int_{-\infty}^{+\infty} h_t(s)x(t-s)ds,$$

where h_t is the impulse response at time t. By interpreting h_t as function of two variables, i.e., $h_t(s) = h(t,s)$ we can write

$$y(t) = \mathbf{H}x(t) = \int_{-\infty}^{+\infty} h(t, t-s)x(s)ds.$$

We denote

$$\sigma(t,\omega) = \int_{-\infty}^{+\infty} h(t,s)e^{-2\pi i\omega s}\, ds,$$

and get

$$\mathbf{H}x(t) = \int_{-\infty}^{+\infty} \sigma(t,\omega)\widehat{x}(\omega)e^{2\pi i\omega t}d\omega.$$

$\mathbf{H} = \mathbf{H}_\sigma$ is a pseudodifferential operator with *Kohn-Nirenberg symbol* σ.

A simple computation gives

$$\mathbf{H}_\sigma x(t) = \iint \widehat{\sigma}(\eta, u) M_\eta T_{-u} x(t) du d\eta$$

where

$$T_u x(t) = x(t-u)$$

$$M_\eta x(t) = x(t) e^{2\pi i \eta t}$$

are the unitary operators of translation and modulation.

$\widehat{\sigma}$ is called the *spreading function* in communication engineering.

Setting

$$F(\omega, x, \tau) := \widehat{\sigma}(\omega, -x) e^{-2\pi i \tau}$$

gives [Howe '80]

$$\pi(F) = \mathbf{H}_\sigma$$

Mobile channels: quantitative analysis

Narrowband communication: signal bandwidth is small compared to carrier frequency (all current mobile comm. systems)

Delay spread: energy of impulse response decays exponentially in time: for fixed s there exist constants $a, c > 0$ such that

$$|h(t,s)|^2 \leq ce^{-a|t|},$$

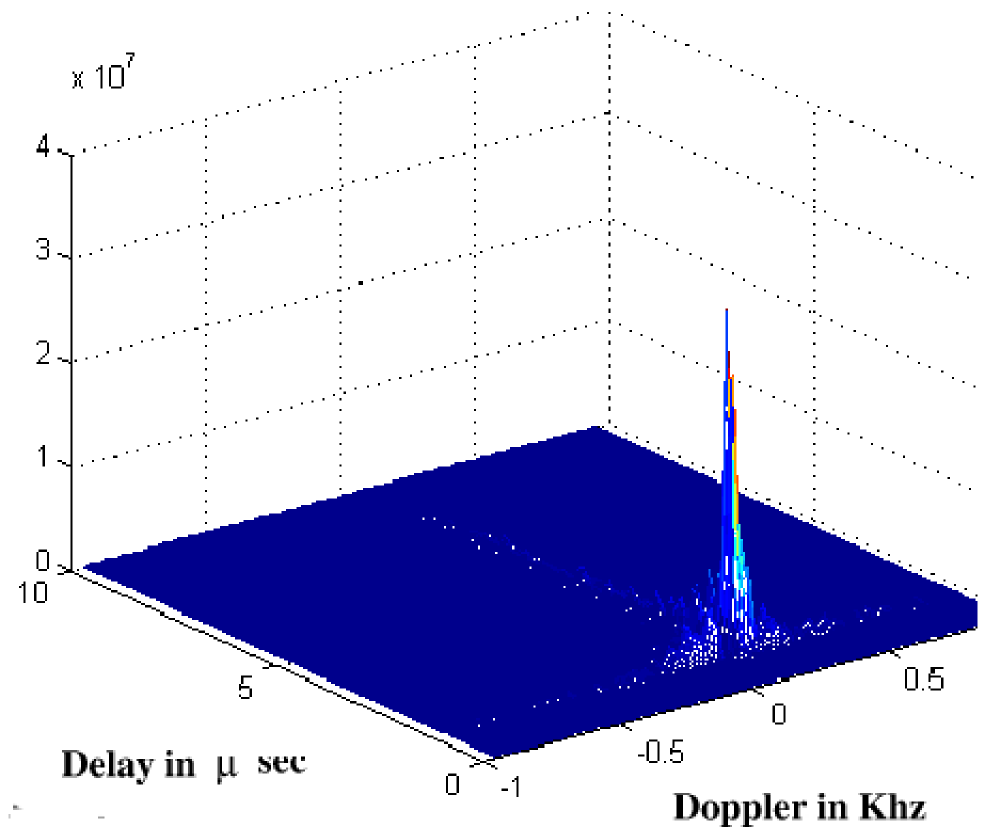
Doppler spread: Doppler shift η_θ is given by

$$\eta_\theta = \frac{v}{\lambda} \cos\theta,$$

where v is velocity v, θ is the angle between direction of moving object and direction of arrival of radio wave, and λ is wave length.

Maximal Doppler shift $\eta_{max} = v/\lambda$. Hence Doppler shift of carrier frequency ω_c is confined to $[\omega_c - \eta_{max}, \omega_c + \eta_{max}]$ and $\widehat{\sigma}(\eta, u)$ has compact support w.r.t. η for fixed u.

Properties of delay spread and Doppler spread imply that spreading function $\widehat{\sigma}$ is localized. (Extreme case: if $\widehat{\sigma}(\eta, u) = \delta_{\eta,u}$ then $\mathbf{H}_\sigma = I$).



Spreading function $\widehat{\sigma}(\eta, u)$

Modulation spaces as symbol classes

Idea: characterize mobile channel $\mathbf{H}_\sigma$ by time-frequency behavior of symbol σ

The *short-time Fourier transform* (STFT) of $f \in \boldsymbol{L}_2(\mathbb{R}^d)$ with respect to the *window* $g \in \mathcal{S}(\mathbb{R}^d)$ is defined by

$$V_g f(t,\omega) = \int_{\mathbb{R}^{2d}} f(s) g(t-s) e^{-2\pi i \omega s} ds, \quad (t,\omega) \in \mathbb{R}^{2d}.$$

Let w be some weight function on $\mathbb{R}^{2d}$. A tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ belongs to the modulation space $M_w^{p,q}(\mathbb{R}^d)$ if

$$\|f\|_{M_w^{p,q}} := \|V_g f(t,\omega)\|_{L_w^{p,q}(\mathbb{R}^{2d})} < \infty$$

Modulation spaces as symbol classes for ΨDO: Tachizawa, Heil, Gröchenig, ...

We will use modulation spaces for two purposes:
(i) as function space for the design of the transmission functions φ_k: $M_w^{1,1}(\mathbb{R})$
(ii) as symbol class for $\sigma(t,\omega)$: $M_w^{\infty,1}(\mathbb{R}^2)$

$M_w^{\infty,1}(\mathbb{R}^2)$ consists of all functions F on $\mathbb{R}^2$ whose norm is defined by

$$\|F\|_{\mathbf{M}_w^{\infty,1}} = \int_{\mathbb{R}^2} \sup_{X\in\mathbb{R}^2} |V_\Psi F(X,\Omega)|\, w(\Omega) d\Omega,$$

with $\Psi \in \mathcal{S}(\mathbb{R}^2)$. For $w \equiv 1$ this is known as *Sjöstrand class*.

Note that $V_\Psi F$ is a 4-dimensional function. In our case $F = \sigma, X = (t,\omega), \Omega = (\eta, u)$

If $\sigma \in \mathbf{M}_w^{\infty,1}(\mathbb{R}^2)$ we write $\mathbf{H}_\sigma \in \mathrm{Op}(\mathbf{M}_w^{\infty,1})$

Theorem:[T.S., 2004] Let $\mathbf{H}_\sigma$ represent a narrowband mobile radio channel. Then $\sigma \in \mathbf{M}_w^{\infty,1}$ where the weight function is given by

$$w(X,\Omega) = w(\Omega) = e^{-|\Omega|^\alpha}, \qquad \text{with } 0 < \alpha < 1.$$

Proof: Result follows from physical characterization of delay spread, Doppler spread, properties of STFT, submultiplicativity of w and link to Beurling algebras.

Having identified proper symbol space we can now study approximate diagonalization of $\mathbf{H}_\sigma$.

We want to find $\{\varphi_k\}_{k\in\mathcal{I}}$ such that matrix $A(\sigma)$ with entries

$$A(\sigma)_{l,k} = \langle \mathbf{H}_\sigma \varphi_k, \varphi_l \rangle, \quad k,l \in \mathcal{I}.$$

is as diagonal as possible for $\sigma \in \mathbf{M}_w^{\infty,1}(\mathbb{R}^2)$.

Gabor systems as transmission functions?

A Gabor system consists of functions $\varphi_{ma,nb}$ of the form

$$\varphi_{ma,nb}(t) = M_{nb}T_{ma}\varphi(t), \;\; m,n \in \mathbb{Z}$$

where $\varphi \in \boldsymbol{L}_2(\mathbb{R})$ is a given *window*, and a, b are the time- and frequency shift parameters. We denote this system by (φ, a, b).

The analysis operator or coefficient operator $C : \boldsymbol{L}_2(\mathbb{R}) \mapsto \ell^2(\mathbb{Z}^2)$ is defined as

$$Cf = \{\langle f, \varphi_{ma,nb}\rangle\}_{n,m \in \boldsymbol{\ell}_2(\mathbb{Z}^2)}.$$

C is just a sampled STFT.

Usually we want φ to be well-localized in time and frequency.
Good choice: $\varphi \in \mathbf{M}^{1,1}_w$ or $\varphi(t) = e^{-t^2}$.

Necessary conditions for invertibility of $A(\sigma)$: $\{\varphi_{ma,nb}\}$ is linearly independent and $\mathbf{H}_\sigma$ is invertible on range(C^*).

To maximize data rate: range(C^*) should be as large as possible, ideally: whole $\boldsymbol{L}_2(\mathbb{R})$.

Balian-Low Theorem: If (φ, a, b) is a Riesz basis for $\boldsymbol{L}_2(\mathbb{R})$, then either $x\varphi(x) \notin \boldsymbol{L}_2(\mathbb{R})$ or $\varphi' \notin \boldsymbol{L}_2(\mathbb{R})$.

Linear independence of (φ, a, b) implies $ab \geq 1$. For completeness of (φ, a, b) in $\boldsymbol{L}_2(\mathbb{R})$ we need $ab \leq 1$ [Perelomov, Daubechies, Rieffel, Seip, Landau, Janssen,...]. Thus for (φ, a, b) to be a Riesz basis for $\boldsymbol{L}_2(\mathbb{R})$ one needs $ab = 1$.

Hence to have a Gabor system (φ, a, b) with good time-frequency localization requires either $ab < 1$ (redundancy) or $ab > 1$ (incompleteness).

A matrix Banach algebra [Baskakov '90]

Let $A = [A_{i,j}]_{i,j\in\mathcal{I}\times\mathcal{I}}$ be a matrix, where $\mathcal{I}$ is some index set. Let w be a weight function, which satisfies the Gelfand-Raikov-Shilov (GRS) condition

$$\lim_{n\to\infty} w(nt)^{\frac{1}{n}} = 1 \qquad \text{for all } t \in \mathbb{R}^d.$$

The Baskakov-Sjöstrand matrix algebra $\mathcal{B}_w$ consists of all matrices A for which

$$\|A\|_{\mathcal{B}_w} := \sum_{k\in\mathcal{I}} \sup_{i-j=k} |A_{i,j}| w(k) < \infty.$$

It is easy to verify that

$$\sum_{k\in\mathcal{I}} \sup_{i-j=k} |A_{i,j}| w(k) = \inf_{a\in\ell^1_w(\mathcal{I})} \{|A_{i,j}| \le a(i-j), i,j \in \mathcal{I}\}.$$

Theorem:[T.S.'04] Assume that (φ, a, b) is an ONS with $\varphi \in \mathbf{M}^{1,1}_w(\mathbb{R})$ and let the weight w satisfy the Gelfand-Raikov-Shilov (GRS) condition

$$\lim_{n\to\infty} w(nr)^{\frac{1}{n}} = 1 \qquad \text{for all } r \in \mathbb{R}^{2d}.$$

If $\mathbf{H}_\sigma \in \mathrm{Op}(\mathbf{M}^{\infty,1}_w)$ and we use $\varphi_{ma,nb}$ as transmission functions then $A(\sigma) \in \mathcal{B}_w$. Furthermore, if $\mathbf{H}_\sigma$ is invertible on range(C^*) then $A(\sigma)^{-1} \in \mathcal{B}_w$.

Proof: Key observations:

$$|\langle \mathbf{H}_\sigma M_\xi T_x \varphi, M_\nu T_y \varphi\rangle| = |V_\Psi \sigma((\xi, y), (x, \xi) - (y, \nu))|$$

where $\Psi = \mathcal{F}^{-1} V_\varphi \varphi$.
Furthermore $\varphi \in M^{1,1}_w(\mathbb{R})$ implies $\Psi \in M^{1,1}_w(\mathbb{R}^2)$.

Important technical detail: use of Wiener amalgam spaces for justification of norm-controlled sampling of STFT when switching to Gabor systems.

Theorem also holds when $ab < 1$, (φ, a, b) is a tight frame and inverse is replaced by pseudoinverse [Gröchenig '04].

General theorem [Gröchenig-T.S. '05]: Let $\mathcal{G}$ be a locally compact abelian group and let the weight satisfy the GRS condition. $\mathrm{Op}(\mathbf{M}_w^{\infty,1}(\mathcal{G}))$ is a Wiener-type Banach algebra: that means, if $\mathbf{H}_\sigma \in \mathrm{Op}(\mathbf{M}_w^{\infty,1}(\mathcal{G}))$ and if $\mathbf{H}_\sigma$ is invertible on $L^2(\mathbb{R})$ then $\mathbf{H}_\sigma^{-1} \in \mathrm{Op}(\mathbf{M}_w^{\infty,1}(\mathcal{G}))$. Furthermore:

$$\mathbf{H}_\sigma \in \mathrm{Op}(\mathbf{M}_w^{\infty,1}(\mathcal{G})) \Longleftrightarrow A(\sigma) \in \mathcal{B}_w$$

(Several special cases due to Sjöstrand 1995, Gröchenig, 2004).

This is one of the rare cases where a symbol space is uniquely characterized by decay properties of matrices, and this even for any lca group!

Orthogonal Frequency Division Multiplexing

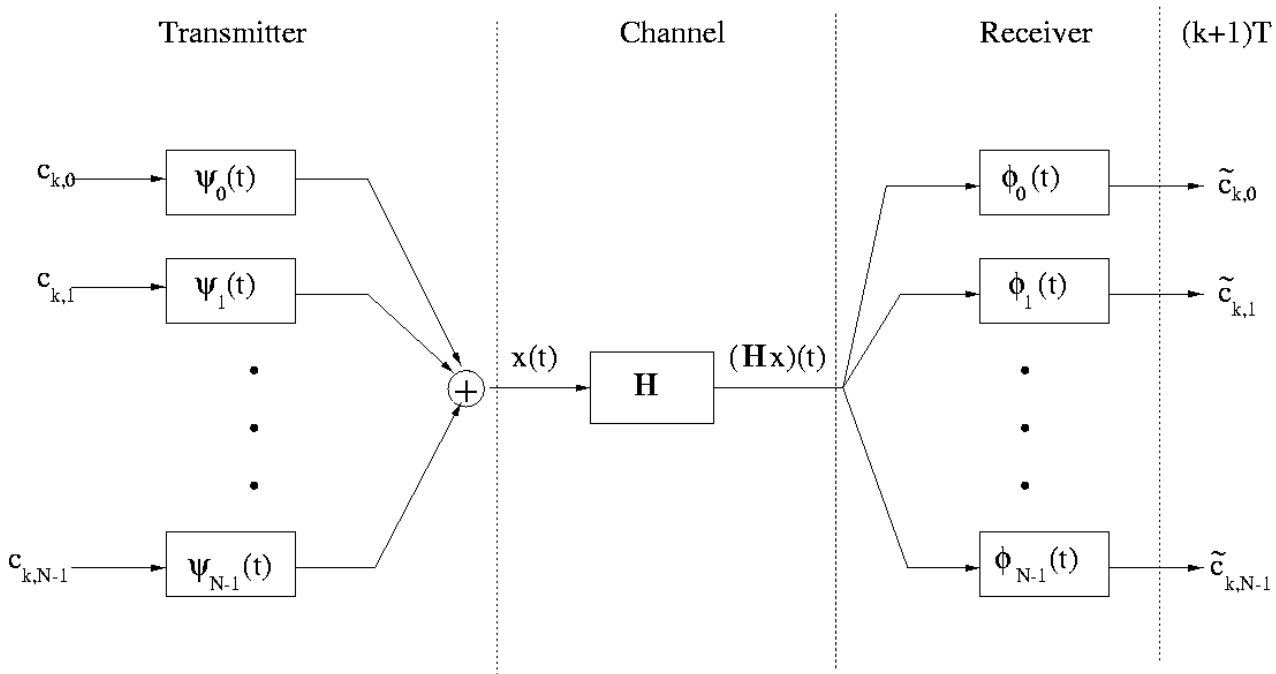
Approximate-diagonalization theorem points to Gabor system as transmission functions for mobile communications/

Engineers already used system similar to a sub-optimal Gabor system in context of *Orthogonal Frequency Division Multiplexing* (OFDM). But using our theoretical framework we can considerably improve existing OFDM systems.

OFDM is used in European digital audio and digital terrestrial TV broadcasting, 3G wireless, IEEE802.11, HSDPA, WIMAX, and probably for UWB.

OFDM

OFDM is a multicarrier system: bandwidth is split into subbands (subcarriers) and a block of data is transmitted simultaneously across subbands.



Assume we split bandwidth Ω into N subbands and let $b = \Omega/N$ (carrier separation). Let $\{c_{k,l}\}_{l=0}^{N-1}$ be the data to be transmitted at time slot k. Let a be the time interval (symbol period) between the transmission of two data blocks. The transmitted OFDM signal is then

$$x(t) = \sum_{k\in\mathbb{Z}} \sum_{l=0}^{N-1} c_{k,l}\varphi(t-ka)e^{2\pi ilbt},$$

where φ is a prototype transmission pulse. Clearly transmission pulses have Gabor-structure

$$\varphi_{ka,lb} = \varphi(t-ka)e^{2\pi ilbt}.$$

Usually φ, a, b are chosen such that $\varphi_{ka,lb}$ are mutually orthogonal. Simple choice that yields orthogonal system (φ, a, b):

$$\varphi = \chi_{[0,a']}, \;\; a' \le a, a = 1/b.$$

This is used in most current OFDM systems. By letting $a' < a$ we effectively insert a *guard interval* between two pulses $\varphi_{ka,lb}$ and $\varphi_{(k\pm 1)a,lb}$

Rectangular pulses plus guard interval is useful in case of multipath but it is a bad choice in presence of Doppler spread.

Reason: if $\varphi = \chi_{[0,a']}$ then $\hat{\varphi} = \text{sinc}_{1/a'}$. While shifted sinc-functions $T_{l/a'}\text{sinc}_{1/a'}$ are orthogonal, a small perturbation caused by Doppler spread completely destroys orthogonality and we get large interference.

Recall: want $A(\sigma) = \{\langle \mathbf{H}_\sigma \varphi_{ka,lb}, \varphi_{k'a,l'b}\rangle\}_{k,l,k',l'}$ to be almost diagonal. But $A(\sigma)$ is only diagonal if there is no Doppler spread, with Doppler $|A(\sigma)_{k,l,k',l'}| \approx 1/((k,l) - (k',l'))$.

Since $\mathbf{H}_\sigma \in$ Op($\mathbf{M}_w^{\infty,1}$) with exponential weight, we should choose $\varphi \in \mathbf{M}_w^{1,1}$, this would result in matrix R whose off-diagonal entries decay like $e^{-\|(k,l)-(k',l')\|^\alpha}$ for some $\alpha < 1$.

Let $\psi(t) := e^{-\pi t^2}$. We know that the functions $T_x M_\omega \psi$ minimize Heisenberg Uncertainty Principle.

But (ψ, a, b) is not an ONS. Want ONS (φ, a, b) such that φ is as close as possible to ψ.

Answer via Gabor duality conditions (= Morita-equivalence of certain C^*-algebras) [Rieffel'88, Daubechies, Landau, Janssen,...] and theorem on tight Gabor frames by Janssen-T.S.,2002.

We say that (φ, a, b) is a *Gabor frame* for $\boldsymbol{L}_2(\mathbb{R})$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{m,n\in\mathbb{Z}} |\langle f, \varphi_{ma,nb}\rangle|^2 \leq B\|f\|^2,$$

for any $f \in \boldsymbol{L}_2(\mathbb{R})$. If $A = B$ the frame is *tight*.

The Gabor frame operator S is defined by

$$Sf = C^*Cf = \sum_{m,n\in\mathbb{Z}} \langle f, \varphi_{ma,nb}\rangle \varphi_{ma,nb}, \ f \in \boldsymbol{L}_2(\mathbb{R}),$$

(g, a, b) is a tight frame if $A = B$, in this case

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, \varphi_{ma,nb}\rangle \varphi_{ma,nb}.$$

Given frame (g, a, b) compute $\varphi = S^{-\frac{1}{2}}g$, then (φ, a, b) is a tight frame.

Theorem:[Janssen-T.S.,JFAA 2002]: Assume (g, a, b) is a Gabor frame, then the function φ_{opt} that minimizes

$$\|g - \varphi\|_2$$

among all functions φ such that (φ, a, b) is a tight Gabor frame, is given by

$$\varphi_{opt} = S^{-\frac{1}{2}} g$$

Since frame operators S for Gabor frames (g, a, b) with window $g \in \mathbf{M}_w^{1,1}$ form a Banach algebra for GRS-weights, Banach square root theorem implies that if $g \in \mathbf{M}_w^{1,1}$ then $S^{-\frac{1}{2}} g \in \mathbf{M}_w^{1,1}$ [T.S.,2002 for L_w^1, Gröchenig 2003 for $\mathbf{M}_w^{1,1}$].

But communications does not need complete, linear dependent tight frames, but (incomplete) linear independent orthonormal systems.

[Rieffel '88]: Morita-equivalence of certain C^*-algebras implies that if (φ, a', b') is a Gabor frame then (φ, a, b) with $a = 1/b', b = 1/a'$ is an ONS for its closed span. (also [Daubechies, Landau, Janssen, Ron, Shen,...])

Thus we can construct OFDM transmission pulses by following method: [T.S.'02/'03]:
1) start with Gabor frame (g, a', b') with $g \in M_w^{1,1}$, e.g. $g(t) = e^{-t^2}$
2) compute $\varphi = S^{-\frac{1}{2}} g$
3) use the set (φ, a, b), $a = 1/b', b = 1/a'$ as OFDM transmission pulses.

Several practical constraints can be included via iterative algorithm. Moreover $S^{-\frac{1}{2}} g$ can be computed very fast. Thus pulse design algorithm can compute optimal pulses in real-time.

Patent pending [T.S, A.Paulraj]

Practical application

Method has been used in collaboration with *Special Communication Systems* to design new modem for short-radio-wave communications.

Pactor III

Landfall Navigation says about Pactor III:
"*It is the most amazing thing we have seen in over 35 years of transmitting data over radio*" (www.landfallnavigation.com/pactor.html).

Further consequences of theory

There are no good results for capacity of time-varying channels.

Our approach gives estimate of capacity in terms of maximal delay spread and maximal Doppler spread.

Crucial difference to time-invariant case: channel estimation should be included in capacity definition! Thus Shannon's capacity definition should be modified for mobile channels.

There are mobile channels that have zero capacity! No information can be transmitted errorfree [T.S.'05, work in progress]

Other results: finite section method

Consider infinite-dimensional system $Ac = d$, where A has a left-inverse.

We want to solve $Ac = d$ numerically and consider the finite system

$$P_n A Q_m c^{(m)} = P_n d$$

where P_n, Q_m are ortho-projections onto n-dim. and m-dim. spaces with $P_n, Q_m \to I$ for $n, m \to \infty$. Does $\|c^{(m)} - c\|_2 \to 0$ for $n, m \to \infty$? How fast is convergence?

Gohberg developed rigorous theory for finite section method for Toeplitz-type matrices

Banach algebra theory provides positive answer for general matrices if $A \in \mathcal{B}_w$. Rate of convergence depends on condition number of A and weight w. [T.S.'02,'04, Gröchenig-T.S.'05]

Outlook

For wideband communications Doppler effect appears as dilation of signals. This suggest to consider noncommutative harmonic analysis on affine group. Thus we should model wireless channel as

$$\mathbf{H}_a x(t) = \iint a(u,s) T_u D_s x(t) du ds$$

where D_s is the dilation operator.

Do wavelets approximately diagonalize such operators? For which symbol spaces? What about symbol calculus? Connection to Fuchs-type operators?

Capacity results for mobile wideband channels?

What we have now: fairly robust, high-data rate wired and fixed wireless communication systems. Based on Wiener and in particular Shannon's theoretical framework "A mathematical theory of communication" [1948]

Underlying mathematics: commutative harmonic analysis, basic probability theory, classical coding theory,...

What we want in the 21st century: reliable high-data rate mobile communication systems.

Need a "mathematical theory for mobile communication"

Underlying mathematics: noncommutative harmonic analysis, advanced probability theory (random matrix theory,...), new coding theory (space-time codes,...),...

"There is nothing so practical as a good theory."

Ludwig Boltzmann