MAT 515– Dr. Z. Ye

FINAL

Fall 2019

Name :________________________________ ZID :________________________________

1. (10 pts) a) Give the definition of $\cos z$ for $z \in \mathbb{C}$ (Either in terms of a power series or exponential functions)

   b) Prove that $\cos z$ is unbounded in $\{z \in \mathbb{C} : |\text{Re}z| \leq 1\}$.

2. (10 points) Let $f(z) = \sum_{n=1}^{\infty} 8^n (z - 2i)^{3n}$. Find the maximum disk where $f(z)$ is analytic, i.e. the radius of the convergence and the convergence disk, AND sketch the disk on the complex plane $\mathbb{C}$.

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3. (10 pts) Let $z = x + iy$. Prove that the function $f(z) = -y + e^{-x}\sin y + i(x + e^{-x}\cos y)$ is analytic in $\mathbb{C}$ AND find $f'(\pi i)$.

4. (10 pts) Evaluate $\displaystyle\int_{|z+i|=2} \left(|z+i|^{2019} + \mathrm{Im}(z)\right) |dz|$.

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5. (20 pts) Find a conformal one-to-one map from the upper-half unit disk $\{z \in \mathbb{C} : \mathrm{Im} z > 0, |z| < 1\}$ to the unit disc $D(0,1)$.

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6. (20 pts) Evaluate $\int_\Gamma \frac{e^z}{(4z-1)\cos(\pi z)} dz$, where:

   a) Γ is the circle $|z| = 1/8$ traversed once counterclockwise.

   b) Γ is the circle $|z| = 2/3$ traversed once counterclockwise.

   c) Γ is the circle $|z| = 1$ traversed once counterclockwise.

7. (15 pts) Let $f(z) = \frac{1}{z^2(e^z+1)}$. (a) Find all isolated singularity of f and their residues.

   (b) Evaluate the integral $\int_{|z|=2\pi} f(z)\, dz$.

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8. (10 pts) (a) State Liouville's theorem for analytic functions.

(b) Let f be an entire function. Prove that f is constant if $f(\mathbb{C}) \cap D(0,\ 1) = \emptyset$.

9. (10 pts) Determine the number of solutions of the equation $z^{10} + 10z + 8 = 0$ in the annulus $1 < |z| < 2$.

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10. (10 pts) (a) State the Identity Principle for analytic functions on region Ω.

(b) If f is analytic on the unit disk $D(0,1)$ and $f(\frac{1}{n}) = 0$ for all positive integers n, then f is identically equal to 0 on $D(0,1)$.

11. (20 pts) Show that $\int_0^{\pi} \frac{d\theta}{2+\cos\theta} = \frac{\pi}{\sqrt{3}}$ by using the Residue Theorem.

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12. (15 pts) Find the Laurent series for the function $f(z) = \dfrac{z}{(z+1)(z-2)}$ in each of the following domains.

(a) $|z| < 1$

(b) $1 < |z| < 2$

(c) $2 < |z| < \infty$

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13. (20 pts) Verify the integral:

$$\int_{-\infty}^{+\infty} \frac{e^{x/2}}{1+e^x}\, dx = \pi.$$

(Hint: You could consider a rectangular contour with height 2π and width $2R$ for any positive real number R.)

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14. (20 pts) **True-False.** If the assertion is true, quote a relevant theorem or reason, or give a proof; if false, give a counterexample or other justification.

   (a) (**T or F**) The point 0 is an isolated singularity of the function $f(z) = (\sin \frac{1}{z})^{-1}$.

   (b) (**T or F**) Let z_0 be an isolated singularity of f. If $\operatorname{Res}(f, z_0) = 0$, then z_0 is removable.

   (c) (**T or F**) The function $\frac{\sin z}{z}$ can be regarded as an entire function.

   (d) (**T or F**) The function $\frac{e^z \sin z}{z^3}$ has a pole at $z = 0$ of degree 3.

   (e) (**T or F**) Let f, g be analytic on the closure of the unit disk $D(0,\ 1)$, i.e. $\overline{D(0,\ 1)}$. If $|f(z) - g(z)| \leq |f(z)| + |g(z)|$ for z with $|z| = 1$, then f and g have the same number of zeros in $D(0,\ 1)$, counting the multiplicities.

   (f) (**T or F**) Let f be analytic on $\overline{D(0,\ 1)}$, i.e., f is analytic in a neighborhood of the unit disk $D(0,\ 1)$. If f has exactly 5 zeros, counting the multiplicities, inside $D(0, 1)$, then $\frac{1}{2\pi i} \int_{|z|=1} \frac{f'(z)}{f(z)}\, dz = 5$.

   (g) (**T or F**) Let $S(z) = \frac{z+i}{z-i}$. The Möbius transformation $S(z)$ maps the real axis to the unit circle.

   (h) (**T or F**) The function $f(z) = z^2$ is conformal in the unit disc $D(0,\ 1)$.