MAT 513– Dr. Z. Ye

Final Exam

Fall 2017

Rules for take-home exam

1 I will turn in the final exam (take-home part) on Monday, 04/30/2018, by 3pm before our final exam takes place in OS2009.

2 I know there is ABSOLUTELY NO EXTENSIONS and late submission will be severely penalized.

3 I write all solutions completely by myself and do not show it or discuss it with anybody.

4 I may use my class notes, textbook, and publicly available materials, including books, online, etc. I may ask Dr. Ye for limited hints.

By signing my name below, I certify that I have read, understand and comply with the above rules.

Name (PRINT) :____________________. Signature :____________________. Date :______________

1. (15 pts) (a) Give the definition of an algebra.

(b) Give the definition of an σ algebra.

(c) Let $X = \mathbb{N}$, the set of natural numbers, and $\mathcal{A} = \{A \subset \mathbb{N} : A \text{ is finite or } A^c \text{ is finite}\}$. Prove that $\mathcal{A}$ is not an σ-algebra. (Hint: Consider $A_n = \{2j : 1 \le j \le n\}$).

2. (10 pts) Let $\mathcal{A}$ be an algebra and $A_n \in \mathcal{A}$, for all $n \in \mathbb{N}$. Then there exists a sequence of sets B_n such that
i) $B_n \in \mathcal{A}$, for all $n \in \mathbb{N}$; ii) $B_n \cap B_m = \emptyset$, if $m \neq n$; iii) $\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} B_n$.

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3. (15 pts) (a) Give the definition of an outer measure on a set X.

(b) Prove that if $(X, \mathcal{A})$ is a measurable space, μ^* is an outer measure on X, $B \in \mathcal{A}$, and we define $\nu^*(A) = \mu^*(A \cap B)$ for $A \in \mathcal{A}$, then ν^* is an outer measure.

4. (15 pts) (a) Define a measure on a measurable space $(X, \mathcal{A})$.

(b) Define a measurable function on a measurable space $(X, \mathcal{A})$.

(c) Suppose $(X, \mathcal{A})$ is a measurable space, f is a real-valued function, and $\{x : f(x) > r\} \in \mathcal{A}$ for each rational number r. Prove that f is measurable.

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5. (20 pts) (a) Let $(X, \mathcal{A}, \mu)$ be a measure space and f is a non-negative function on X. Define $\int f d\mu$. (Hint: You have to start with measurable characteristic functions, then simple functions, and so on.)

(b) Rigorously **state** and **prove** Fatou's Lemma.

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6. (10 pts) Find the limit

$$\lim_{n\to\infty} \int_0^n \left(1+\frac{x}{n}\right)^{-n} \log\left(2+\cos\frac{x}{n}\right) dx$$

and justify your reasoning (clearly state the convergence theorem you may use).

7. (10 pts) Let f be integrable on $(X, \mathcal{A}, \mu)$, $f > 0$ a.e., and $A \subset X$ a measurable set. If $\int_A f = 0$, prove that $\mu(A) = 0$.

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8. (15 pts) Let $\{f_n\}$ be a sequence of non-negative integrable functions such that $\sum_{n=1}^{\infty} f_n = f$ is integrable. Prove that

$$\int \Big(\sum_{n=1}^{\infty} f_n\Big) d\mu = \sum_{n=1}^{\infty} \int f_n d\mu.$$

9. (20 pts) Let $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ and

$$f(x) = \begin{cases} 0, & \text{if } x \in A \cap [0.\ 1], \\ xe^{x^2}, & \text{if } x \in [0,\ 1] \setminus A. \end{cases}$$

Prove that f is measurable, Riemann integrable on $[0, 1]$, and find, if any,

$$(R) \int_0^1 f(x)\, dx \qquad \text{and} \qquad (L) \int_{[0,\ 1]} f(x)\, dx$$

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10. (20 pts) (a) Let f be a function on an interval $[a, b]$. Define that f is of bounded variation on $[a, b]$.

(b) If f is of bounded variation on $[a, b]$, then f can be written as a difference of two increasing functions on $[a, b]$.

11. (15 pts) (a) Let f be a function on an interval $[a, b]$. Define that f is absolutely continuous on $[a, b]$.

(b) If f' is bounded on an interval $[a, b]$, prove that f is absolutely continuous on $[a, b]$.

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12. (35 pts) True-False. If the assertion is true, quote a relevant theorem or reason, or give a proof; if false, give a counterexample or other justification.

   (a) Let $A_{2n} = [0,1)$ and $A_{2n+1} = [1,2)$. Then $\liminf_{n\to\infty} A_n = \{0\}$.

   (b) Let $E \subset \mathbb{R}$ and m be the Lebesgue measure on $\mathbb{R}$. If $m(E) = 0$, then E is countable.

   (c) Suppose that f is a bounded measurable function in an interval I, then f is continuous at least at one point in the interval I.

   (d) Let A be a subset in $\mathbb{R}$, then χ_A is a measurable function.

   (e) If f is a measurable function on $[0,\ 1] \subset \mathbb{R}$, then for every $\delta > 0$, there are a continuous function g and a closed set $F \subset [0,1]$ such that $m([0,\ 1] \setminus F) < \delta$ and $f(x) = g(x)$ for $x \in F$, where m is the Lebesgue measure.

   (f) Let $f_n \geq 0$ be a sequence of integrable functions. If $\lim_{n\to\infty} f_n = f$, then

$$\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n = \int f.$$

(g) Let f be integrable, and $\mu(A) = 0$. Then $\int_{X \setminus A} f = \int_X f$.

(h) Let $A = \mathbb{Q}$, $B = \mathbb{R} \setminus \mathbb{Q}$ and $f = \chi_A - \chi_B$. Then f is Riemann integrable on the interval $[0, 1]$.

(i) Any bounded variation function is differentiable almost everywhere.

13. (**Bonus** 10 pts). Let $I = [a, b]$ and m be the Lebesgue measure. Suppose f is a positive increasing function in I, and put $M = \sup_{x \in I} f(x)$ and $E = \{x \in I : \ f(x) = M\}$. Prove

a) $\lim\limits_{n \to \infty} \int_I \left(\frac{f(x)}{M} \right)^n dx = m(E)$.

b) $\lim\limits_{n \to \infty} \int_I f'(x) \left(\frac{f(x)}{M} \right)^n dx = 0$.

Hint: $m(E) = \int \chi_E$.