MAT 335– Dr. Z. Ye FINAL FALL 2016

Name :______________________________ ID # :______________________________

1. (15 pts) Given the linear system $\begin{cases} \quad\;\; x_2 \quad\quad\;\; +2x_4 = 0 \\ x_1 \quad\quad +x_3 \quad\quad\;\; = -1 \\ 2x_1 + x_2 \quad\quad\;\; -x_4 = 1. \\ \quad\;\; 5x_2 \quad\quad\;\; +10x_4 = 0 \end{cases}$

   (a) Write the linear system in a matrix equation.

   (b) Find its augmented matrix.

   (c) Find the reduced row echelon form for the augmented matrix.

   (d) Find all solutions, if any, and express your solutions in the vector form.

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2. (15 pts) Let n be a positive integer and A an $n \times n$ matrix.

   (a) State the definition of linear independence of vectors $\mathbf{w_1}, \mathbf{w_2}, \cdots, \mathbf{w_k}$.

   (b) Prove that if $\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_k}$ are linearly independent vectors in $\mathbb{R}^n$ and A is invertible, then $A\mathbf{u_1}, A\mathbf{u_2} \cdots, A\mathbf{u_k}$ are linearly independent.

   (c) Let $\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_k}$ be linearly independent vectors in $\mathbb{R}^n$. Are $A\mathbf{u_1}, A\mathbf{u_2} \cdots, A\mathbf{u_k}$ always linearly independent? Justify your answers.

3. (10 pts **bonus**) If $\mathbf{w_1}, \mathbf{w_2}, \cdots, \mathbf{w_k}$ is an orthogonal set of non-zero vectors in $\mathbb{R}^n$, prove that $\mathbf{w_1}, \mathbf{w_2}, \cdots, \mathbf{w_k}$ is linearly independent.

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4. (20 pts)

   (a) Define span$(\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_k})$, where $\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_k}$ are vectors in $\mathbb{R}^n$.

   (b) Define the Column Space and the Null Space of a matrix A.

   (c) Find a basis for Col(A) and Nul(A), where $A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 3 & 2 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$.

   (d) Verify The Rank Theorem: i.e. Rank A + dim (Nul A)$= n$.

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5. (10 pts) Let $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 3 & 0 \\ 2 & -1 & 0 & 2 \\ -1 & 0 & 1 & 0 \end{bmatrix}$ be a 4×4 matrix and let k be any positive integer and $B = \begin{bmatrix} 1 & 2 & 3 & \cdots & k \\ 0 & 2 & 3 & \cdots & k \\ 0 & 0 & 3 & \cdots & k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & k \end{bmatrix}$ a $k \times k$ matrix. Find $\det(A)$ and $\det(B)$.

6. (10 pts) Let x be a real number and $\mathbf{u} = \begin{bmatrix} -1 \\ x \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$.

(a) Find a value x such that $||2\mathbf{u} + \mathbf{v}|| = 1$.

(b) Find a value x such that two vectors $\mathbf{u}, \mathbf{v}$ are orthogonal.

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7. (20 pts) Let $A = \begin{bmatrix} 4 & -1 & -2 \\ 0 & 3 & -2 \\ 0 & 0 & 4 \end{bmatrix}$.

   (a) Find the characteristic polynomial of A.

   (b) Find all eigenvectors of A.

   (c) Find the eigenspace of A for each eigenvalue.

   (d) Diagonalize the matrix A, i.e. write $A = PDP^{-1}$, if possible.

   (e) Let P be in (d). Calculate $A^{2016}P =$.

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8. (24 pts) Let A, P be $n \times n$ matrices. Mark the following statements True or False. Justify each of your answers by citing a theorem or giving a counter-example.

(a) If A is diagonalizable, then A is invertible.

(b) If A is diagonalizable, then A has n distinct eigenvalues.

(c) If $\mathbb{R}^n$ has a basis of eigenvectors of A, then A is diagonalizable.

(d) If $AP = PD$, with D diagonal, then the non-zero columns of P must be eigenvectors of A.

(e) Every $n \times n$ matrix A is diagonalizable.

(f)If A is diagonalizable, then A is orthogonally diagonalizable.

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9. (25 pts) Let $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$.

   (a) Prove $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent.

   (b) Use the Gram-Schmidt process to produce an orthogonal basis for $\mathbf{W} = \text{span}(\mathbf{u}, \mathbf{v}, \mathbf{w})$.

   (c) Find an orthonormal basis for $\mathbf{W}$ in (b).

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10. (25 pts) Let $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$.

    (a) Define a least squares solution of a linear system $A\mathbf{x} = \mathbf{b}$.

    (b) State the normal equation of $A\mathbf{x} = \mathbf{b}$ in terms of $\hat{\mathbf{x}}$.

    (c) Find a least squares solution of $A\mathbf{x} = \mathbf{b}$ by solving the normal equation in (b) for $\hat{\mathbf{x}}$.

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11. (26 pts) Orthogonally diagonalize the matrix $A = \begin{bmatrix} 3 & 4 \\ 4 & 9 \end{bmatrix}$. That means to find an orthogonal matrix U and a diagonal matrix D such that $A = UDU^T$.

(a) Prove A is a symmetric matrix.

(b) State the definition of an orthogonal matrix.

(c) Find all eigenvalue of A.

(d) Find all eigenvectors for each eigenvalue.

(e) Define an orthogonal matrix U and a diagonal matrix D such that $A = UDU^T$.

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12. (10 pts) Prove that if matrices A and B are similar, then A and B have the same characteristic polynomials.

    You will get partial credits if you state the following definitions:

    (a) The similarity of two matrices A, B.

    (b) The characteristic polynomial of A.