MATH 311– Dr. Z. Ye **Final Exam** SPRING 2017

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1. (15 pts) (a) State the $\epsilon - N$ definition of a convergent sequence $\{x_n\}$, say, $\lim_{n\to\infty} x_n = A$.

   (b) Prove $\lim_{n\to\infty} \frac{3n^2}{n^2+1} = 3$ by using the $\epsilon - N$ definition.

2. (20 pts)

   (a) State the definition of a continuous function f at c.

   (b) State the definition of a differentiable function f at c.

   (c) Let $f(x) = x \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Prove f is continuous at 0.

   (d) Prove that f is NOT differentiable at 0.

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3. (10 pts) Let $\{x_n\}$ be a sequence of real numbers. Suppose that $\{x_n\}$ converges to L . Prove that $\{|x_n|\}$ converges to $|L|$ by using the $\epsilon - N$ language.

4. (20 pts)

   (a) State the Mean Value Theorem.

   (b) Let $n \geq 1$ be an integer and let $b \geq a \geq 0$ be any positive real numbers. Prove that

$$na^{n-1}(b-a) \leq b^n - a^n \leq nb^{n-1}(b-a).$$

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5. (20 pts)

(a) State the definition of a uniformly continuous function f on an interval I.

(b) Prove that $f(x) = 1/x$ is NOT uniformly continuous on the interval $(0, 1]$.

(c) Prove that $f(x) = 1/x$ is uniformly continuous on the interval $(\frac{1}{100}, 1]$.

6. (10 pts) Find the limit $\lim_{x\to 0} \frac{e^{\sin x + e^x}(2x - 3\sin x)}{(3x + \sin x)\cos(x + e^x \sin x)} =$

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7. (25 pts)

a) State the definition of an absolutely convergent series.

b) State the definition of a conditionally convergent series.

c) Prove that $\sum_{n=4}^{\infty} \frac{(-1)^n}{n \log^2 n}$ (Hint: the integral test) absolutely converges.

d) Prove that $\sum_{n=4}^{\infty} \frac{(-1)^n}{n \log n}$ conditionally converges.

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8. (10 pts) Find all points in $\mathbb{R}$ where the series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n}$ converges.

9. (10 pts) Let $f(x) = x^2$ and ${}^tP$ a tagged partition of $[0, 1]$ is $\{0, [0, 1/3]; 1/2, [1/3, 2/3]; 1, [2/3, 1]\}$. Find the Riemann sum $S(f, {}^tP)$ over the interval $[0, 1]$ with the tagged partition ${}^tP$.

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10. (10 pts) Prove Theorem 5.12, i.e. prove that if f is decreasing on an interval $[a, b]$, then f is Riemann integrable on $[a, b]$ by using the $\epsilon - \delta$ language.

11. (20 pts) Let

$$f_n(x) = \begin{cases} 1, & \text{if } x \leq 0, \\ n, & \text{if } 0 < x < \dfrac{1}{n} \\ 1/2, & \text{if } x \geq \dfrac{1}{n}. \end{cases},$$

a) State the definition of a convergent sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ on $\mathbb{R}$.

b) Sketch the graphs of $f_1(x), f_2(x), f_3(x)$ on $\mathbb{R}$.

c) Find the limit of the function sequence $\{f_n(x)\}$ on the interval $(-\infty, \infty)$.

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12. (30 pts) True-False. If the assertion is true, quote a relevant theorem, axiom or reason, or give a proof; if false, give a counterexample or other justification.

(a) The series $\sum_{k=1}^{\infty} a_k$ converges if and only if the sequence $\{a_k\}$ converges.

(b) $\sum_{n=2}^{\infty} \frac{1}{n(n+1)} = 2.$

(c) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} |a_n|$ diverges.

(d) If $\lim_{n\to\infty} a_n = 0$ and $a_n > 0$ for all n, then $\sum_{n=1}^{\infty} a_n$ converges.

(e) $\lim_{n\to\infty} x^n = 0$ for all $x \in [0, 1]$.

(f) $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$ for $x \in [0, 1)$.

(g) If f is a continuous function on an interval I, then f is differentiable on I.