Simple Harmonic Motion

Introduction

A primary motivation for studying simple harmonic motion is its general applicability to a variety of diverse branches of physics. An example of an elementary system which exercises simple harmonic motion is a mass attached to a spring under the influence of gravity. Consider an object suspended from a spring in the presence of gravity, as depicted in figure 1. If the spring obeys Hooke's Law (An interesting and informative discussion of Hooke's Law can be found in the Wikipedia encyclopedia, Hooke's law.), then

$$f = -kx \ , \tag{1}$$

where x is the displacement of the spring from its unstretched position, f is the restoring force of the spring, and k is a constant independent of the displacement. According to Newton's Second Law the equation of motion satisfied by the object is

$$ma = mg - kx \ , \tag{2}$$

where m is the mass of the object, a is its acceleration, and g is the acceleration of gravity. The coordinate system has been chosen so that the direction of the positive x–axis is down. Consider the object displaced by an amount A from its equilibrium position and released from rest. Such an object undergoes simple harmonic motion. It can be shown that Equation 2 can be solved to obtain the displacement x' of the object from equilibrium,

$$x' = A\cos\left(\frac{2\pi}{T}t\right) \ , \tag{3}$$

t being the amount of time elapsed after the object is initially displaced. The equilibrium displacement x_0 of the object is

$$x_0 = \frac{mg}{k} \ , \tag{4}$$

so that

$$x' = x - x_0 \ . \tag{5}$$

The period of oscillation T satisfies the relationship

$$T^2 = \frac{4\pi^2}{k}(m + m_e) \ , \tag{6}$$

where the effective mass of the spring m_e is related to the mass of the spring m_s by

$$m_e = \kappa m_s \; , \tag{7}$$

κ, ($0 \leq \kappa \leq 1$), being a quantity which depends on the mass distribution of the spring along its length. The spring used in this experiment is uniformly tapered from one end to the other so that its diameter (and mass density) varies constantly from one end to the other. It can be shown that

$$\kappa = \frac{1}{6}\left(\frac{d_t + 3d_b}{d_t + d_b}\right) \; , \tag{8}$$

where d_b is the the diameter of the spring at the free end (the bottom), and d_t is the diameter at its fixed end (the top).

In this experiment varying amounts of mass are attached to a uniformly tapered spring and the extent to which their motion can be described as simple harmonic is assessed.

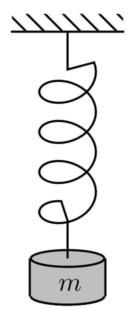

Figure 1: Simple Harmonic Motion

Procedure

The equipment used in performing the experiment is shown in Figures 2 and 3.

1. Measure the mass of the spring m_s, the mass of the weight holder m_h, and the diameters of the spring at either end, the wider end d_t and the narrower end d_b. Record their values in Table 1.

2. Attach the spring to the support structure as shown in Figure 1, the wider end connected to the support structure. Attach the weight holder to the free end of the spring.

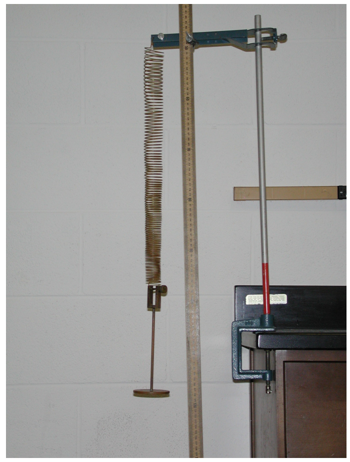

Figure 2: The equipment includes a uniformly tapered spring whose end of smaller diameter is free to oscillate, a mass holder, and a measuring stick, two meters long.

3. Select a convenient location on the weight holder and note the value of its position $x = x_h$ on the meter stick. In Table 2 this value of x corresponds to $\Delta m = 0$. Record its value in the Table.

4. Add .100 kg to the weight holder so that $\Delta m = .100$ kg. Record its equilibrium position x in Table 2. Continue adding .100 kg increments of mass and measuring its equilibrium position in this manner until .500 kg have been added to the weight holder. Record each position x in the Table.

5. Remove all but one of the .100 kg masses. Measure the amount of time for the mass to complete 10 oscillations, as follows. Displace the weight holder approximately $A = 10$ cm from its equilibrium position, and release it. Using the stop watch measure the time for the weight holder to complete 10 oscillations, and record its value in Table 3 in the column labeled $10 \times T$. Perform the same time measurement for the remaining mass increments Δm, recording their values in the table. According to Equations 3 and 6 the period of oscillation is independent of the displacement A so that the displacements need not be precisely 10 cm. This assertion is true to the extent that the force exerted by the spring obeys Hooke's Law. i.e. the elastic limit of the spring is not exceeded.

Analysis

Numerical results should be reported to the correct number of significant figures.

1. Complete Table 2, using $\Delta x = x - x_h$ and $f = \Delta m\, g$, where $g = 9.80$ m/s^2, the acceleration of gravity.

Figure 3: Additional equipment used in the experiment includes vernier calipers, five 100 gram masses, and a stop watch.

2. Complete Table 3, using $m = \Delta m + m_h$. The period T is obtained by dividing the quantities in the column labeled $10 \times T$ by ten.

3. Using Microsoft Excel plot a graph of f vs. Δx (Graph 1). Here f is the ordinate, and Δx is the abscissa. Perform a linear regression, setting options so that the intercept is required to be zero and that the regression equation and estimated value of r^2 are displayed.

4. Using Microsoft Excel plot a graph of T^2 vs. m (Graph 2). Here T^2 is the ordinate, and m is the abscissa. Also, perform a linear regression, setting options so that the regression equation and estimated value of r^2 are displayed. Do not require the intercept to be zero in this case.

5. From Graph 1 obtain the estimates of the slope M_{est} and r^2. Using Eq. 14 calculate δM. According to Eq. 1 M_{est} is an estimate of the spring constant k_{est}, i.e. $k_{\text{est}} = M_{\text{est}}$. The quantities σ_y and σ_x can be evaluated using the Microsoft Excel library function STDEV. According to Eq. 1 $\delta k = \delta M$. Report $k_{\text{est}} \pm \delta k$ in Table 4.

6. From Graph 2 obtain the estimates of the slope M_{est}, the intercept B_{est}, and r^2. Using Equations 11 and 12 calculate δM and δB. The quantity $\bar{x}$ can be calculated using the Microsoft Excel library function AVERAGE. In order to relate the regression parameters to physical characteristics of the spring, re–express Equation 6 as follows,

$$T^2 = \frac{4\pi^2}{k} m + \frac{4\pi^2}{k} \kappa m_s \,. \tag{9}$$

Estimate the spring constant and its standard error using $k_{\text{est}} = \frac{4\pi^2}{M_{\text{est}}}$ and $\delta k \approx k_{\text{est}}\frac{\delta M}{M_{\text{est}}}$. Report $k_{\text{est}} \pm \delta k$ in Table 4. Estimate κ_{est} and $\delta\kappa$ using $\kappa_{\text{est}} = \frac{B_{\text{est}}}{M_{\text{est}} m_s}$ and $\delta\kappa \approx \kappa_{\text{est}}\left[(\frac{\delta M}{M_{\text{est}}})^2 + (\frac{\delta B}{B_{\text{est}}})^2\right]^{\frac{1}{2}}$. Report $\kappa_{\text{est}} \pm \delta\kappa$ in Table 4.

7. Calculate the theoretical value of κ using Eq. 8. Using the criterion Eq. 16 determine if the theoretical and experimental values of κ are consistent.

8. The spring constant k has been estimated by two different methods. The two methods are considered to yield consistent values if their confidence intervals given by Eq. 16 overlap. Determine if the two methods yield consistent estimates of the spring constant.

Appendix

Consider two sets of data y_i and x_i ($i = 1 \ldots N$) where the y_i are assumed to be linearly related to the x_i according to the relationship

$$y_i = Mx_i + B + \epsilon_i \; . \tag{10}$$

The ϵ_i is a set of uncorrelated random errors. Estimates of the slope M_{est} and y–intercept B_{est} can be obtained using a variety of techniques, such as the method of least squares estimation. Furthermore, it can be shown that the standard error of the slope δM is

$$\delta M = \sqrt{\frac{1-r^2}{N-2}}\frac{\sigma_y}{\sigma_x} \; , \tag{11}$$

and the standard error of the intercept δB is

$$\delta B = \sigma_y \sqrt{\frac{1-r^2}{N-2}} \sqrt{\frac{N-1}{N} + \frac{\bar{x}^2}{\sigma_x^2}} \; , \tag{12}$$

where r is the estimated correlation coefficient between the y_i and x_i, σ_y and σ_x are the estimated standard deviations of the y_i and x_i, and $\bar{x}$ is the estimated average of the x_i.[1] The quantities $\bar{x}$, σ_x, and σ_y can straightforwardly be calculated using function keys on a scientific calculator or defined functions in Excel. If, in the regression equation, Eq. 10, the quantity B, the y-intercept, is required to be zero, i.e. the regression equation is now

$$y_i = Mx_i + \epsilon_i \; , \tag{13}$$

then the degrees of freedom in the random errors are $N-1$ rather than $N-2$ so that

$$\delta M = \sqrt{\frac{1-r^2}{N-1}}\frac{\sigma_y}{\sigma_x} \; . \tag{14}$$

[1] Note: in Equation 12 σ_x and σ_y are unbiased estimates.

To reflect the statistical uncertainty in a quantity Q, where Q is either the slope M or y-intercept B, the quantity Q is typically reported as

$$Q_{\text{est}} \pm \delta Q \, , \tag{15}$$

which can be understood informally to mean that with high probability the true value of Q lies within the interval

$$[Q_{\text{est}} - 1.96\,\delta Q, Q_{\text{est}} + 1.96\,\delta Q] \, , \tag{16}$$

and its best estimate is Q_{est}. [2] In general, any quantity expressed in the form of Eq. 15 can be interpreted according to Eq. 16.

[2]The seemingly arbitrary value 1.96 has a precise statistical interpretation. For sufficiently large N it can be shown that the statistical distribution of Q_{est} is a normal distribution of mean Q (the true value of Q) and standard deviation δQ. Thus, the probability that the true value of Q lies outside of the interval given by Equation 16 is .05.

m_h [kg]	m_s [kg]	d_t [mm]	d_b [mm]

Table 1: Data I

Δm [kg]	x [m]	Δx [m]	f [N]
0			
.100			
.200			
.300			
.400			
.500			

Table 2: Data and Calculations I

Δm [kg]	$10 \times T$ [s]	T [s]	m [kg]	T^2 [s^2]
.100				
.200				
.300				
.400				
.500				

Table 3: Data and Calculations II

	Graph 1	Graph 2	Theory
$k_{\text{est}} \pm \delta k$			
$\kappa_{\text{est}} \pm \delta\kappa$			

Table 4: Results