The Cavendish Experiment

1 Introduction

In 1687 Newton published the *Principia* in which he presented a mathematical description of the gravitational force. In Newton’s theory of gravitation the gravitational attraction, $F$, exerted by one object on another is expressed according to the relationship

$$ F = G \frac{m_1 m_2}{R^2}. \quad (1.1) $$

The quantities $m_1$ and $m_2$ are the masses of two objects whose spatial separation is $R$. The value of the constant of proportionality, $G$, was not known to Newton and, in fact, was not accurately determined until the latter half of the nineteenth century. The calculation of its value was based on the results of an experiment to determine the density of the earth performed by Henry Cavendish, and published in 1798\(^1\). The purpose of this experiment is to perform a modern version of the Cavendish experiment, determine the gravitational constant, $G$, and compare it to its accepted value.

2 Theory

The primary apparatus used to perform this experiment is the torsion balance which is shown in Figure 1. The torsion balance is depicted in Figure 2 with relevant components labeled for illustrative purposes. Initially, the internal boom is at rest. Then the two large masses are put in place and attract the two smaller masses causing a torque to be exerted on the internal boom so that the it rotates from its initial point of equilibrium. As the boom rotates, the thin wire which is fixed at the top twists until the wire exerts a counter torque on the boom. If the rotational kinetic energy is dissipated, the boom will come to rest at a new point of equilibrium so that the following relationship for the torques is satisfied:

$$ 2Fd = k\theta_D. \quad (2.1) $$

The term on the right of the equality is the counter torque resulting from the wire. The quantity $k$ is the torsion constant of the wire, and $\theta_D$ is the magnitude of the angular displacement between the two equilibrium points. The term on the left of the equality is the torque resulting from the gravitational attraction. There is a factor of 2 because each of the large masses exerts a force $F$ (Eq. 1.1) on a smaller mass. The quantity

\(^1\)An informative and interesting discussion of Newton’s Theory of gravitation and the Cavendish experiment can be found in the Wikipedia \[5\],[4] and references therein.
$d$ is the distance between a small mass and the wire, i.e. the axis of rotation. The large masses are positioned, initially, so that the force exerted by each large mass acts perpendicularly to the boom. In practice, $\theta_D \ll 1$ so that the resulting torque is approximately constant. Substituting Eq. 1.1 into Eq. 2.1 and solving for the gravitational constant we obtain

$$G = \frac{R^2 k \theta_D}{2m_1 m_s d}.$$  \hspace{0.2cm} (2.2)

The experimental values $m_l, m_s$, and $R$ can be determined directly while determination of the torsion constant, $k$, and the equilibrium angle, $\theta_D$ require special consideration.

The value of the torsion constant can be obtained as follows. If the internal boom is allowed to oscillate freely in the absence of the large masses, the boom undergoes damped, simple harmonic motion. The damping results primarily from air dragging against the boom as it oscillates and not from friction internal to the wire or at the location where the wire is attached at the top of the apparatus. Thus, the time dependent angle of rotation of the boom satisfies the equation of motion

$$I \frac{d^2 \theta}{dt^2} = -k \theta - b \frac{d \theta}{dt} + F_d,$$  \hspace{0.2cm} (2.3)

The quantity $I$ is the moment of inertia of the boom including the small masses; $\theta$ is the angle through which the boom rotates; $b$ is a constant which characterizes the velocity dependent dissipative force. Equation 2.3 can be solved for the angular coordinate, $\theta(t)$, of the boom as a function of time $t$

$$\theta(t) = \theta_e + e^{-bt} A \cos \left( \frac{2\pi}{T} (t - \delta) \right),$$  \hspace{0.2cm} (2.4)

where $A$ is the angular displacement of the boom at time $t = \delta$, $\theta_e$ is the equilibrium angle, and $T$ is the period of oscillation. The torsion constant satisfies the relationship

$$k = I \left( \frac{4\pi^2}{T^2} + \frac{b^2}{4I} \right).$$  \hspace{0.2cm} (2.5)

The torsion constant can be evaluated from Equation 2.5 using values of parameters on the right side of the equation, either measured directly or obtained from observing the boom oscillating freely.

Determining the angle, $\theta_D$, although straightforward in principle, is difficult in practice. This method involves, first, positioning the large mass to the right, as shown in Figure 1 allowing the oscillations of the internal boom to damp, and then and measuring the equilibrium angle. Then, the large mass is positioned on the left and the process is repeated. Depending whether the large mass is placed on the left side or the right side of the boom, the equation of motion satisfied by the angular coordinate, $\theta_L$ or $\theta_R$, of the boom is

$$I \frac{d^2 \theta}{dt^2} = -k \theta - b \frac{d \theta}{dt} \pm 2F_d,$$  \hspace{0.2cm} (2.6)

where $2F_d = k \theta_D$ (See Eq. 2.1). The plus (minus) sign is associated with the left side
The solutions, $\theta_L(t)$ and $\theta_R(t)$ of Eq. 2.6 are

$$\theta_L(t) = (\theta_e + \theta_D) + e^{-bt} A \cos\left[\frac{2\pi}{T} (t - \delta)\right]$$

$$\theta_R(t) = (\theta_e - \theta_D) + e^{-bt} A \cos\left[\frac{2\pi}{T} (t - \delta)\right]$$

(2.7) (2.8)

After a sufficiently long time, i.e. after the oscillations have damped, the angle $\theta_D$ is then calculated from the measured values of the equilibrium angles of the boom. Specifically,

$$\theta_D = \lim_{t \to \infty} \frac{\theta_L(t) - \theta_R(t)}{2}$$

(2.9)

The primary difficulty is that even though the oscillations of the boom damp because of air viscosity, the boom drifts unpredictably without reaching static equilibrium. The drifting has been attributed to anelastic forces resulting from strains within the thin wire \[3, 2\]. In order to mitigate the effect of drifting we apply an alternative method of determining $\theta_D$. In this method the internal boom is allowed to oscillate. Then, when the boom reaches a turning point (either an angular maximum or minimum), the large mass is positioned to the right (See Figure 1) or to the left, respectively. As a consequence, the boom is driven alternately toward the right or left equilibrium angles, $\theta_e \mp \theta_D$. Consider, for example, the case where the internal boom oscillates and $N$ consecutive turning points, $\theta_1, \theta_2, \cdots \theta_N$, ($N$ an odd integer) are observed. Furthermore assume that the boom is driven at $\theta_1$ and $\theta_2$, etc. but not at $\theta_N$. Then, using Eq’s 2.7 and 2.8 one can derive the following relationships \[1\]

$$\theta_2 = (\theta_e \mp \theta_D) - x(\theta_1 - (\theta_e \mp \theta_D))$$

$$\theta_3 = (\theta_e \pm \theta_D) - x(\theta_2 - (\theta_e \pm \theta_D))$$

$$\vdots$$

$$\theta_N = (\theta_e \pm \theta_D) - x(\theta_{N-1} - (\theta_e \pm \theta_D))$$

(2.10)

where

$$x = e^{-bT/2}.$$  

(2.11)

In Eq’s 2.10 the top sign applies if $\theta_1$ is a maximum, and the bottom sign applies if it is a minimum. Solving each of the Eq’s 2.10 for $\theta_D$ one obtains

$$\theta_D = (-1)^{(1+1)/2}(-1)^1 \left[\frac{\theta_2 - \theta_e + x(\theta_1 - \theta_e)}{1 + x}\right]$$

$$\theta_D = (-1)^{(1+1)/2}(-1)^2 \left[\frac{\theta_3 - \theta_e + x(\theta_2 - \theta_e)}{1 + x}\right]$$

$$\vdots$$

$$\theta_D = (-1)^{(1+1)/2}(-1)^n \left[\frac{\theta_{n+1} - \theta_e + x(\theta_n - \theta_e)}{1 + x}\right]$$

$$\vdots$$

$$\theta_D = (-1)^{(1+1)/2}(-1)^{N-1} \left[\frac{\theta_N - \theta_e + x(\theta_{N-1} - \theta_e)}{1 + x}\right]$$

(2.12)
where \( 1 \leq n \leq N - 1 \). Averaging the calculated values of \( \theta_D \) in Eq. 2.12 one obtains an estimate of \( \theta_D \)

\[
\theta_D = \frac{|(1 - x)(\theta_1 - \theta_2 + \theta_3 - \theta_4 + \cdots + \theta_N) - \theta_1 + x\theta_N|}{(N - 1)(1 + x)}.
\]

Because of the factors \((-1)^n\) in Eq. 2.12 no terms involving \( \theta_e \) appear when the average is calculated.

Figure 1: The Cavendish balance. Shown is the Cavendish apparatus without the large masses in position. The large masses are placed on the external boom which can be rotated to either the left or right for the purpose of driving the internal boom counterclockwise or clockwise.

3 Procedure

All data and calculations should be recorded in scientific notation with the correct number of significant figures.

1. Measure the mass of each large ball. Calculate the average mass, and record it in Table 1. Measure the diameter of each large ball. Calculate the average diameter, and record it in Table 1.

2. Transcribe the data from the link Cavendish apparatus corresponding to your apparatus to Table 1. These are the data specific to each apparatus. Their relationship to the apparatus is shown in Figure 3. In addition, the value of the parameter \( \text{scale} \) is included in the file. The rotation angle of the internal boom is measured by an electronic sensor which requires calibration. The parameter \( \text{scale} \) is used for this purpose. A detailed description of the calibration procedure is given in the user manual.
3. Rotate the external boom so that it is perpendicular to the plane of the glass enclosure (See Figure 1). Carefully (disturbing the apparatus as little as possible), place the large masses in the holes on the external boom.

4. Execute the program `copy_data_files.bat`.

5. Execute the program `Cavendish USB`. Set the Sampling rate to 10 Hz, and the Number of points to 16384 (27 min 18.4 sec). Using the mouse, activate the record button in the lower left corner of the window. Using the zoom button (the cursor becomes a magnifying glass) for the ordinate (labelled y), magnify the plotted ordinate of the boom. Reset the cursor using the arrow button (The cursor becomes an arrow.). When the plot reaches a turning point, rotate the external boom so that it almost touches the glass case of the apparatus (Be careful not to disturb the apparatus, unduly, or to allow the large mass to strike the glass enclosure.). If the turning point is a relative maximum, rotate to the right; if it is a relative minimum, rotate to the left. Continue in this manner, alternating the boom from one side to the other, for a total of eight turning points. Using the mouse measure the ordinates of the eight turning points, and record them in Table 2. When the boom reaches the ninth turning point record its value in the Table, also. The external boom does not need to be rotated at the ninth turning point. Save the data in the file `data_mass.txt`. Rotate the external boom until it is perpendicular to the plane of the glass and remove the large masses, being careful not to disturb the apparatus.

6. Observe the plot in the display window. As soon as the plot approximates a damped harmonic form, activate the stop button and reset button. When the ordinate approaches a relative maximum, activate the record button. Using the
mouse measure the abscissae of the first two relative maxima, and record their values in Table 3. Allow the plot to evolve until three relative maxima and three relative minima are displayed. Save the data in the file `data_no_mass.txt`. Exit the program `Cavendish USB`.

7. Transcribe the data from Tables 1, 2, and 3 to the files named in the table captions. In the subdirectory `sample_data` are examples of the corresponding relevant files with sample data entered.

8. Open the Maple program `cavendish.mw`. Activate the `Edit` drop-down menu. Activate the menu item `Execute`, followed by `Worksheet`. This series of commands execute the program. The output from the program comprises three parts: selected input data, calculated results, and two plots. The plot in red is the angular data of the freely oscillating boon, i.e., without the large masses driving the oscillations. The plot in green is a regression based on the theoretical model Eq. 2.4. Inspect the plots. The regression plot should fit the data extremely well. If the fit is not good, check that the data entered into file `input_maxima.txt` from Table 3 are correct. In the program $\theta_D$ is evaluated using Eq. 2.13 with $N = 9$, the estimated regression parameters and input from file `input_turning_points.txt`. The gravitational constant, $G$, is evaluated using 2.1, estimated regression parameters, and other input data. The estimated value of $G$ is listed with the results. Record the appropriate results in Table 4.

9. Using Eq’s 1.1, A.1 calculate the gravitational force, $F$, that the small mass exerts on the large mass. An accurate value for $G$ can be obtained from the Wikipedia online encyclopedia. In addition, calculate the weight, $W$, of the large mass, i.e. the gravitational force that the earth exerts on the large mass. In calculating the weight use $W = mg$, where $g = 9.80$. Also, calculate the ratio $F/W$. Record these results in Table 5. The purpose of these calculations is to give some appreciation of how sensitive is the torsion balance to extremely small forces.

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2The regression model includes a term linear in time to account for drift.
Figure 3: Cavendish apparatus (top view). The outer rectangle represents the outer structure of the apparatus when viewed from above. The interior rectangle denotes the inner boom on which are positioned the small masses. Relevant dimensions are labelled.

### Appendix

Some parameters appearing in equations presented in Sections 1 and 2 are not directly measurable. Here, we relate these parameters to those parameters which are directly measurable.

The separation, $R$, between the large mass and the small mass (See Eq. 1.1) is approximated

$$ R \approx \frac{w_c + d_l}{2}, \quad \text{(A.1)} $$

where $d_l$ is the average diameter of a large mass, and $w_c$ is the average width of the Cavendish apparatus at the two locations where the small masses are positioned on the internal boom (See Figure 3). The moment arm $d$ (See Eq. 2.1 and Figure 2) is

$$ d \approx \frac{d_b - d_s}{2}, \quad \text{(A.2)} $$

where $d_s$ is the average diameter of a small mass, and $d_b$ is the maximum separation between the small masses, measured between the surfaces of the two masses (See Figure 3).

The moment of inertia, $I$, of the combination of internal boom and the two small masses is given by

$$ I \approx I_{\text{boom}} + 2 \left( \frac{2}{5} \right) m_s \left( \frac{d_s}{2} \right)^2. \quad \text{(A.3)} $$

The moment of inertia of the boom is

$$ I_{\text{boom}} \approx \frac{1}{12} m_s (l_b^2 + w_b^2), \quad \text{(A.4)} $$

where $l_b$ and $w_b$ are the length and width of the boom (See Figure 3).
References

http://people.uncw.edu/olszewski/phy101lab/laboratory/manual.pdf


<table>
<thead>
<tr>
<th><strong>Cavendish Apparatus</strong> –</th>
<th></th>
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</thead>
<tbody>
<tr>
<td>avg. mass of large ball, ( m_l ), [kg]</td>
<td>avg. diameter of large ball, ( d_l ), [m]</td>
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<tr>
<td>avg. mass of small ball, ( m_s ), [kg]</td>
<td>avg. diameter of small ball, ( d_s ), [m]</td>
</tr>
<tr>
<td>avg. width of Cavendish balance, ( w_c ), [m]</td>
<td>distance between outer surface of small balls, ( d_b ), [m]</td>
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<tr>
<td>mass of boom, ( m_b ), [kg]</td>
<td>length of boom, ( l_b ), [m]</td>
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<tr>
<td>width of boom, ( w_b ), [m]</td>
<td>angular calibration factor, ( S )</td>
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Table 1: Input parameters for the file *input_parameters.txt*. 
Table 2: Input parameters for the file `input_turning_points.txt`.

<table>
<thead>
<tr>
<th>$i$</th>
<th>ordinate of turning point, $\theta_i$, [mrad]</th>
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<tbody>
<tr>
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Table 3: Input parameters for the file `input_maxima.txt`.

<table>
<thead>
<tr>
<th>abscissa of first maximum, $\delta_1$, [s]</th>
<th>abscissa of second maximum, $\delta_2$, [s]</th>
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Table 4: Comparison between experimental and theoretical values of the gravitational constant.

<table>
<thead>
<tr>
<th>experimental value of the gravitational constant, $G_{exp}$</th>
<th>theoretical value of the gravitational constant, $G_{theo}$</th>
<th>percent error</th>
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The force that small mass exerts on large mass, $F$ [N], is compared to the weight of large mass, $W$ [N], in Table 5.

<table>
<thead>
<tr>
<th>force that small mass exerts on large mass, $F$ [N]</th>
<th>weight of large mass, $W$ [N]</th>
<th>$F/W$</th>
</tr>
</thead>
</table>

Table 5: A comparison of the gravitational force exerted by the small mass on the large mass with the force exerted by the earth on the large mass, i.e. the weight of the large mass.