From baking a cake to solving the diffusion equation

Edward A. Olszewski[a)]
Department of Physics, University of North Carolina at Wilmington, Wilmington, North Carolina 28403-5606

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We explain how modifying a cake recipe by changing either the dimensions of the cake or the amount of cake batter alters the baking time. We restrict our consideration to the génoise and obtain a semiempirical relation for the baking time as a function of oven temperature, initial temperature of the cake batter, and dimensions of the unbaked cake. The relation, which is based on the diffusion equation, has three parameters whose values are estimated from data obtained by baking cakes in cylindrical pans of various diameters. The relation takes into account the evaporation of moisture at the top surface of the cake, which is the dominant factor affecting the baking time of a cake.

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I. INTRODUCTION

To cook and do it well requires little knowledge of the sciences in general and physics in particular. Indeed, most cookbooks are just that—cookbooks: one merely follows the steps in a recipe and obtains the finished product. Usually no discussion of the physical principles on which a recipe is based is given, making it difficult in many instances to modify the recipe in nontrivial ways. There are, however, noteworthy exceptions.[1–4]

Surprisingly, there appears to be a dearth of research related to the cake baking process. Much of what appears in the literature is specifically related to bread baking.[5,6] A notable exception is the work of Lostie and collaborators.[7,8] They analyzed the baking process of sponge batter during the first baking period.[9,10] As described in Ref. 7 the data used in their analysis were obtained from controlled experiments in which the cake batter was heated only from the top, with the sides and bottom of the pan thermally insulated. Based on their experiments they proposed a one-dimensional model of the baking process to predict the spatial dependence and time evolution of water content (water and vapor), temperature, gas phase pressure, and porosity (the proportion of non-solid volume to total volume) of the cake batter. The model has nine adjustable parameters whose values are determined by fitting the model to their data.

The motivation for this article is to provide an answer to the question of how the baking time of a cake varies when its recipe is modified, either by changing the dimensions of the cake or the amount of cake batter. To make the problem manageable we have restricted our analysis to one type of cake only, the génoise, one of the basic cakes of classic French cuisine. Although the technique for baking this cake is somewhat exacting, the ingredients are readily available, facilitating attempts to reproduce our results. The theoretical basis of our solution is the diffusion equation.

The outline of the article is as follows. First, we briefly discuss the quantitative aspects of cake baking. Next, we describe the experimental techniques and data used in the subsequent analysis. We, then, propose a simple model for the cake baking process and use it to obtain a semiempirical formula for estimating the baking time of the génoise.

II. CAKE BAKING AS A DIFFUSION PROCESS

A. Cake baking from a quantitative point of view

Before modeling the cake baking process, we first make more precise the imprecise measuring techniques that have customarily accompanied the instructions for baking a cake. A typical cake recipe lists the ingredients with amounts, a description of the technique used to prepare the cake batter, and the baking time for the recipe. Until recently, the amounts of the ingredients have generally been given in units of liquid measure, such as the cup or tablespoon. We can surmise that the reason for these units has been the scarcity of mass measuring scales in the home. The reason for using dry measure (measuring by scale) rather than liquid measure is to increase the likelihood of consistently producing a cake with the desired taste, texture, and appearance. Generally, the time necessary to bake a cake is accompanied by some subjective measure of determining whether the cake has baked to the proper degree of doneness. Such subjective measures include when a toothpick thrust into the center of the cake comes out clean, when the cake shows a faint line of shrinkage from the sides of the pan, when the top springs back lightly, and when you can smell the cake. To a first approximation these criteria can be quantified by equating the degree of doneness to the temperature at the center of the cake, although we can imagine more complex criteria that include other measurable quantities such as the temperature gradient.

B. The génoise

The recipe for the génoise is adapted from that described by Pépin.[11–13] In Table I we converted the units to dry measure. Qualitatively, we have found that the recipe prepared using traditional measuring techniques, that is, proportions of ingredients are measured using measuring cups (liquid measure) rather than a scale (dry measure), produces a cake consistent in texture, taste, and appearance with one prepared using dry measuring techniques. Quantitatively, the amounts measured and re-measured using traditional techniques typically differ by less than 2% compared to amounts measured by weight.

According to the recipe the cake batter is placed in two cake pans, 8 in. in diameter by 1.5 in. deep, filling each pan 3/4 full. The cakes are baked in an oven of temperature $T_b = 350\ ^\circ\mathrm{F}$ for a time between 22 and 25 min. To establish

Table I. The proportion of ingredients for the génoise is given in traditional measure as well as dry measure.

	Traditional measure	Dry measure (g)
Eggs	6 large	298
Sugar	3/4 cup	176
Vanilla extract	1/2 tsp	2
All purpose flour	1 cup	144
Butter	3/4 stick	114

benchmarks for this study we have prepared the recipe, with exceptions as noted, filling two 8 in. cake pans to a depth of 1 in. We assessed the degree of doneness using traditional techniques[14] and observed the baking time to be approximately 17 min. The qualitative measure of doneness corresponds to a temperature $T_f=203\ ^\circ\mathrm{F}$ at the center of the cake. The recipe was prepared several times with baking times varying by no more than 2 min.

C. Theory from a naive perspective

We initially believed that we could model the cake baking process as a simple diffusion process.[15] Although this model is inadequate, it is the basis of the final model and is thus presented. This model is based on the diffusion equation,[16]

$$\mathcal{D}\nabla^2 T = \frac{\partial T}{\partial t}, \tag{1}$$

where $T=T(t,r,\theta,z)$ is the temperature of the cake at time t at the position (r,θ,z) (in cylindrical coordinates) within the cake, and $\mathcal{D}$ is the heat diffusivity (assumed constant) of the cake batter.

We assume the cake batter to be in a cylindrical pan of radius R and thickness Z at initial temperature T_i and baked in an oven of constant temperature, T_b. We solve the diffusion equation with these initial conditions for the temperature T, which is independent of θ because of azimuthal symmetry,

$$T(t,r,z) = T_b + (T_i - T_b)\frac{2R\sqrt{Z}}{\pi}\sum_{n=1,m=1}^{\infty}\frac{[1-(-1)^n]}{n x_{m,0}}\Phi_{mn}, \tag{2}$$

where the Φ_{mn} are the normalized eigenfunctions of the diffusion equation,

$$\Phi_{mn} = \frac{2}{RJ_1(x_{m,0})\sqrt{Z}}\sin\left(\frac{n\pi z}{Z}\right)\frac{J_0\left(\frac{x_{m,0}r}{R}\right)}{x_{m,0}J_1(x_{m,0})} \times \exp\left[-\left\{\left(\frac{n\pi}{Z}\right)^2 + \left(\frac{x_{m,0}}{R}\right)^2\right\}\mathcal{D}t\right], \tag{3}$$

and $x_{m,0}$ is the mth root of the zeroth-order Bessel function.[17] In Eq. (2) the first term on the right-hand side is the steady state solution. The second term is the solution particular to the boundary conditions and vanishes at the surface bounding the cylinder. Only the Bessel functions of zeroth order appear because of azimuthal symmetry. The other factors are required to reproduce the initial temperature in the interior of the cake at $t=0$.

The initial temperature of the cake batter $T_i \approx 80\ ^\circ\mathrm{F}$ corresponds to the temperature at which the cake batter is prepared. The initial and final temperatures of the cake and temperature of the oven satisfy the relation

$$\frac{T_f - T_i}{T_b - T_i} < \approx 1. \tag{4}$$

For temperatures T_f that satisfy Eq. (4), the infinite series in Eq. (2) evaluated at the center of the cake can be approximated by a single term

$$T(t_f,0,Z/2) = T_f = T_b + (T_i - T_b)F, \tag{5}$$

where

$$F = \begin{cases} \frac{8}{\pi x_{1,0}J_1(x_{1,0})}\exp\left[-\left\{\left(\frac{\pi}{Z}\right)^2 + \left(\frac{x_{1,0}}{R}\right)^2\right\}\mathcal{D}t_f\right] & (C_1 \leqslant 1 \text{ and } C_2 \leqslant 1)) \\ \frac{4}{\pi}\exp\left[-\left(\frac{\pi}{Z}\right)^2\mathcal{D}t_f\right] & (C_1 > 1) \\ \frac{2}{x_{1,0}J_1(x_{1,0})}\exp\left[-\left(\left(\frac{x_{1,0}}{R}\right)^2\right)\mathcal{D}t_f\right] & (C_2 > 1). \end{cases} \tag{6}$$

Here

$$C_1 = \frac{2}{x_{1,0}J_1(x_{1,0})}\exp\left[-\left(\frac{x_{1,0}}{R}\right)^2\right)\mathcal{D}t_f\right], \tag{7a}$$

$$C_2 = \frac{4}{\pi}\exp\left[-\left(\frac{\pi}{Z}\right)^2\mathcal{D}t_f\right] \tag{7b}$$

and t_f is the amount of time for the temperature of the center of the cake to become equal to T_f.

This simplification to Eq. (2) is most readily verified by direct calculation. We can solve Eq. (5) explicitly for the baking time, t_f,

$$\frac{1}{t_f} \propto \begin{cases} \frac{x_{1,0}^2}{R^2} + \frac{\pi^2}{Z^2} & (C_1 \leq 1 \text{ and } C_2 \leq 1) \\ \frac{\pi^2}{Z^2} & (C_1 > 1) \\ \frac{x_{1,0}^2}{R^2} & (C_2 > 1), \end{cases} \tag{8}$$

where $x_{1,0} \approx 2.40$ is the first root of the zeroth-order Bessel Function.

The constant of proportionality depends only on the unknown heat diffusivity constant $\mathcal{D}$, which can, in principle, be estimated from the baking time of a given recipe. In any case, if the baking time is known for a cake of specified dimensions, then Eq. (7) can be used to estimate the baking time of a cake of arbitrary dimensions.

For practical applications it is useful to express Eq. (8) with its numerical constants appearing explicitly:

$$\frac{1}{t_f} \propto \begin{cases} \frac{2.34}{D^2} + \frac{1}{Z^2} & (C_1 \leq 1 \text{ and } C_2 \leq 1) \\ \frac{1}{Z^2} & (C_1 > 1) \\ \frac{2.34}{D^2} & (C_2 > 1), \end{cases} \tag{9}$$

where D is the diameter of the cake.[18,19] We use Eq. (9) to estimate the baking time of a cake with a 2.1 in. radius and depth of 4.0 in. The calculation assumes the same initial conditions as for the preparation of the basic recipe described previously.[20] The predicted baking time is ≈85 min in comparison to the actual baking time, based on an internal temperature of 203 °F is 26 min.

III. EXPERIMENT

A. Procedure

To understand why the previously described model of the cake baking process is inadequate, we collected data for different amounts of cake batter baked in cylindrical cake pans of various dimensions. Seven cakes were baked with radii ranging from 2.1 to 6.5 in. and depths from 1.0 to 4.0 in.; the baking times ranged from 17 to 42 min. The data consist of the mass of each cake before and after baking, the diameter and depth of the cake, the depth of the cake after baking, and the temperature at the center of the cake recorded at 1 min intervals until the cake's internal temperature reaches 203 °F.[21] The experimental setup for recording the internal temperature of the cake is shown in Fig. 1.

In most ovens the temperature of the oven is measured by a probe that is close to the oven walls and relatively distant from whatever is being baked in the oven. The temperature at the oven walls is typically higher than the temperature at the surface of what is being baked. In this study temperatures were monitored next to the cake pan, with the heating element of the oven regulated manually to maintain the desired oven temperature. By monitoring the oven temperature close to the surface of the cake we minimized convective effects, making conduction the primary mechanism of heat transfer. After the cake is baked, the depth and mass was remeasured.

The baking time of an individual cake is very sensitive to the depth of the cake batter. The reason is that the batter is relatively shallow, as is typical of cakes of this type. For example, if the cake batter is approximately 1.5 in. deep, an error in measurement of $\frac{1}{8}$ in. is a 12% error. Even when using extreme care to make depth measurements at the center of the cake, it is difficult to ensure that the depth remains constant across the surface of the cake. This problem is exacerbated for the cakes of larger diameter used in this study. The fact that cakes rise while baking is not a problem, but identical cake recipes baked on different occasions do not rise precisely the same way. These uncertainties explain why for many cake recipes there is a range of baking times of 15% or more. Given the sensitivity of the baking time to the depth of the cake batter, it seems reasonable to expect more variation in the baking time of cakes of larger radii because cohesion of the cake batter between regions separated by large distances would be suppressed causing variations in the depth at these locations to be independent.

Fig. 1. The oven, baking pan, and two thermocouple sensors and temperature measuring instruments. One sensor monitors the internal temperature of the cake, and the other the temperature of the oven.

There is another practical consideration that is a potential source of error. Because of the shallow depth of the batter relative to the size of the temperature probe, it is difficult to ensure that the probe is inserted accurately into the center of the cake. If extremely careful experiments are needed for making accurate theoretical predictions, then our results would be of little use. Because it was impractical to automate the data collection process, specifically the temperature/time measurements, the data collection was tedious.

In Figs. 2 and 3 we show plots of the temperature measured at the center of two cakes as a function of time, one cake of radius 4.0 in. radius and depth 1.0 in. and another cake of 2.1 in. radius and depth 4.0 in. We also show the theoretical fits to the data based on Eq. (2). The summations in Eq. (2) have been truncated at $n, m = 17$.[22] As is apparent in the figures, the model based on the diffusion equation does not account for the features of the data.

B. Revised model

We can make several qualitative and quantitative observations from an analysis of our data. For a baked cake, not only does the volume of the cake increase, but the mass of the cake decreases.[23] With hindsight it was naive to believe that a model based only on Eq. (2) could account for the features of the data. The cake batter is a viscous liquid at the start of the baking process and a solid at the end. Thus, not only

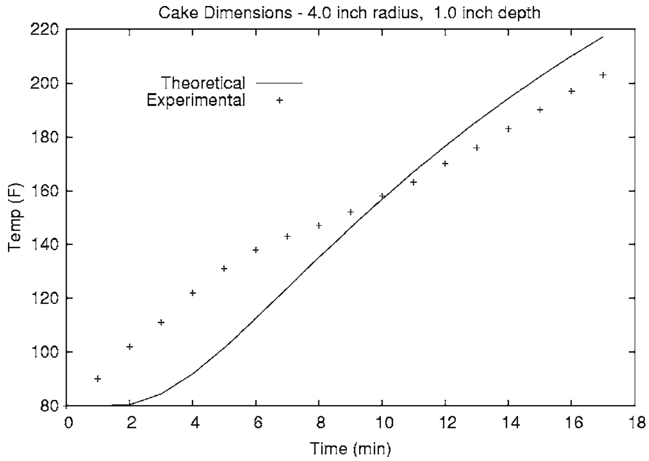

Fig. 2. The temperature of a génoise with a 4.0 in. radius and 1.0 in. depth recorded at 1 min intervals is shown along with the theoretical fit based on Eq. (2).

conduction, but also convection and even radiation contribute to the baking process. In addition, the parameters associated with these processes of heat transfer, that is, the thermal conductivity, density, specific heat, heat diffusivity, and convection coefficient, depend on temperature to some extent. In any case, assuming that the heat diffusivity is constant as in Eq. (2) is simplistic. To improve the model we assume that the thermal conductivity of the cake batter varies during the baking process, primarily because of moisture evaporation at the top cake surface.

In the revised model moisture evaporates at the top surface of the cake causing the cake batter to dry out so that the heat diffusivity decreases. The cake batter consists of two homogeneous mixtures, each with its own heat diffusivity. In Fig. 4 we diagrammatically represent the cake baking process. As baking occurs, the interface between the two homogeneous mixtures of cake batter (dry on top and moist on the bottom) moves from the top to the bottom of the pan. The quantity a denotes the interface between the two mixtures at an arbitrary time and $\mathcal{D}$ and $\mathcal{D}'$ are the heat diffusivity of the dry and moist mixtures, respectively. To reduce the time for numerical computations we assume that the cake batter remains

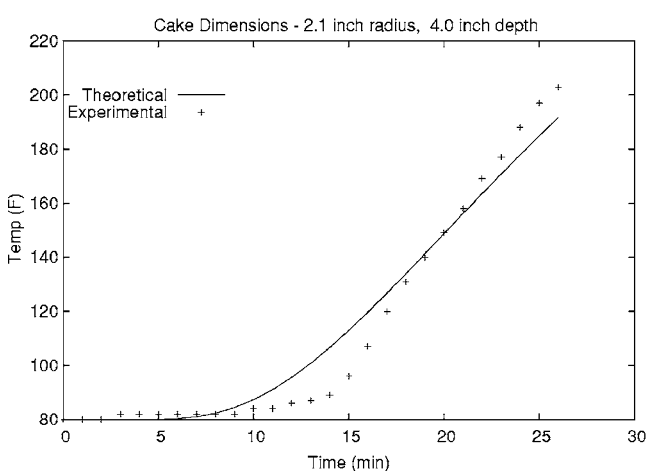

Fig. 3. The temperature of a génoise with a 2.1 in. radius and 4.0 in. depth recorded at 1 min intervals along with the theoretical fit based on Eq. (2).

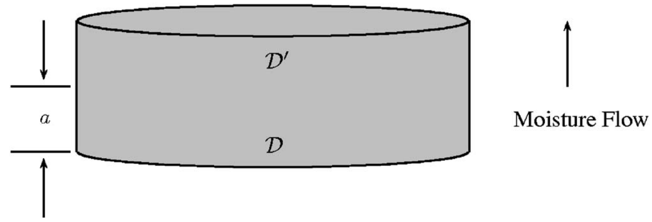

Fig. 4. As the cake is baked, moisture evaporates through the top surface of the cake resulting in a two component system, one with a thermal diffusivity $\mathcal{D}$, corresponding to the moist batter in the bottom portion of the pan and another with thermal diffusivity $\mathcal{D}'$, corresponding to the dry batter in the top portion of the pan.

uniformly moist in time interval $0 \leq t \leq t_1$. In the time $t_1 < t \leq t_2$ the cake comprises two homogeneous mixtures, dry batter in the top half and moist batter in the bottom half of the pan, that is, the interface remains fixed at $a = Z/2$. At time t_2 and until it completes baking at time t_f (interval 3), the cake is uniformly dry.[24]

For time $t \leq t_1$ and $t_2 < t \leq t_f$ the evolution of the temperature is calculated using the eigenfunctions of the diffusion equation given in Eq. (2), substituting $\mathcal{D}$ or $\mathcal{D}'$, as appropriate. For $t_1 < t \leq t_2$ we require eigenfunctions obtained from solving the diffusion equation for two contiguous, homogeneous media whose interface is located at $z = a$.

The unnormalized eigenfunctions Ψ_{mj} with eigenvalue E_{mj} are given by

$$\Psi_{mj} = \exp(-E_{mj}t) J_0\left(\frac{x_{m,0} r}{R}\right) \mathcal{Z}_{mj}(z), \tag{10}$$

where

$$\mathcal{Z}_{mj}(z) = \begin{cases} A_{mj} \sin(\lambda_{mj} z)(z) & (0 \leq z \leq a) \\ B_{mj} \sin(\lambda'_{mj}(Z - z)) & (a < z \leq Z). \end{cases} \tag{11}$$

The coefficients A_{mj} and B_{mj} are determined by the normalization condition and the boundary conditions at $z = a$. The quantities λ_{mj} and λ'_{mj} are related by

$$E_{mj} = \mathcal{D}\left[\lambda_{mj}^2 + \left(\frac{x_{m,0}}{R}\right)^2\right] = \mathcal{D}'\left[\lambda_{mj}'^2 + \left(\frac{x_{m,0}}{R}\right)^2\right], \tag{12}$$

and satisfy the eigenvalue equation

$$\frac{\tan(\lambda_{mj} a)}{\mathcal{D}\lambda_{mj}} = -\frac{\tan(\lambda'_{mj}(Z - a))}{\mathcal{D}'\lambda'_{mj}}. \tag{13}$$

The eigenvalue equation is obtained from the boundary conditions at the interface separating the two homogeneous mixtures of cake batter:

$$\mathcal{Z}(z)\big|_{z=a^-} = \mathcal{Z}'(z)\big|_{z=a^+}, \tag{14a}$$

and

$$\mathcal{D}\frac{\partial \mathcal{Z}(z)}{\partial z}\bigg|_{z=a^-} = \mathcal{D}'\frac{\partial \mathcal{Z}'(z)}{\partial z}\bigg|_{z=a^+}. \tag{14b}$$

Equation (14b) is the result of applying the continuity equation at the interface. We take $\mathcal{D} > \mathcal{D}'$ because the moist cake batter thermally conducts better than the dry cake batter.

As a function of λ'_{mj}, we have

Table II. A summary of the data used in this study. Lengths are in inches, time in minutes, and $\mathcal{D}$ in in.^{2}/min. A representative sample of these data was selected for modeling purposes. The values M and N have been selected to ensure that the theoretical curves are relatively smooth. The adjustable parameters have been determined by trial and error.

			Adjustable parameters				Accuracy	
Radius	Depth	t_f	$\mathcal{D}$	$\mathcal{D}'$	t_1	t_2	M	N
4.0	1.0	17	0.029	0.0035	2	2	10	10
2.1	4.0	26	0.012	0.047	10	∞	10	10
2.1	2.0	20	0.025	0.010	1	15	10	10
5.0	1.8	41	0.035	0.005	2	2	10	10
6.5	1.6	35	⋯	⋯	⋯	⋯	⋯	⋯
5.0	1.0	20	⋯	⋯	⋯	⋯	⋯	⋯
2.1	1.8	19	⋯	⋯	⋯	⋯	⋯	⋯

$$\lambda_{mj}^2 = \frac{\mathcal{D}'}{\mathcal{D}}\lambda_{mj}'^2 - \left(\frac{\mathcal{D}-\mathcal{D}'}{\mathcal{D}}\right)\left(\frac{x_{m,0}}{R}\right)^2. \tag{15}$$

The solutions of the eigenvalue equation have the following properties: $\lambda_{mj}^2 \leqslant \lambda_{mj}'^2$; there are no solutions for $\lambda_{mj}'^2 < 0$; the derivative of an eigenfunction is discontinuous at $z=a$; and some eigenfunctions are oscillatory in the region $0 \leqslant z \leqslant a$ and decaying in the region $a < z \leqslant Z$.

We now prescribe the recipe for implementing the revised model.

1. Specify values for the parameters $\mathcal{D}$, $\mathcal{D}'$, t_1, and t_2.
2. Substitute the temperature of the oven T_b, the initial temperature T_i of the cake, the radius R of the cake pan, and the depth Z of the cake, as well as the value of $\mathcal{D}'$ into Eq. (2) to obtain the internal temperature of the cake $T(t)$ for $0 \leqslant t \leqslant t_1$. $T(t_1)$ becomes the initial temperature of the temperature distribution for $t_1 < t \leqslant t_2$.
3. Expand $T(t_1)$ in terms of the eigenfunctions, Ψ_{mj} Eq. (10), to obtain the temperature $T(t)$ during the interval $t_1 < t \leqslant t_2$. $T(t_2)$ becomes the initial temperature of the temperature distribution for $t_2 < t \leqslant t_f$.
4. Expand $T(t_2)$ in terms of the eigenfunctions Φ_{mn}, Eq. (3), to obtain the temperature $T(t)$ during the interval $t_2 < t \leqslant t_f$. Substitute $\mathcal{D}'$ for $\mathcal{D}$ in Φ_{mn}.

The evaluation of the infinite series for the temperature in the various time intervals requires that each series be truncated at a finite number of terms, depending on the degree of accuracy desired. For $0 \leqslant t \leqslant t_1$ assume that Eq. (2) is terminated for integers M $(m \leqslant M)$ and N $(n \leqslant N)$. To achieve approximately the same degree of accuracy in the interval $t_2 < t \leqslant t_f$, we apply the same values to the corresponding infinite series. For $t_1 < t \leqslant t_2$, the temperature is expanded using Ψ_{mj} also for values of $m \leqslant M$, but the selection of the truncation for j is more complicated, and all values of j such that

$$\lambda_{mj} \leqslant \frac{(N+1/2)\pi}{Z-a} \tag{16}$$

are used. We note that solving for all eigenvalues λ_{mj} is nontrivial because there is not a one to one correspondence between those in the interval $t_1 < t \leqslant t_2$ and those in the other intervals.

We have used these expansions to determine the temperature at the center of the cake for four representative, sets of data. We summarize the results in Table II and Fig. 5.[25] The data have been selected to span the range of cake dimensions. As is apparent in Fig. 5 it is possible to obtain a reasonable fit to the data by varying $\mathcal{D}$, $\mathcal{D}'$, t_1, and t_2. Equally good fits have been obtained for the other three sets of data.

The primary purpose of these models has been to provide insight into the baking process so that quantitative estimates can be obtained for the baking time of cakes of various dimensions. These results are consistent with the qualitative aspects of the revised model. In accordance with Table II, cakes of larger depth, although taking longer to bake, have an effective thermal diffusivity that is greater than cakes of smaller depth. Cakes of smaller radius have an effective thermal diffusivity that is greater than cakes of larger radius.

For cakes of less depth moisture can evaporate from the top surface of the cake relatively quickly causing the cake to dry out leading to a decrease in the thermal diffusivity because of the lack of moisture content. For an amount of baking time t, we hypothesize that the moisture content within the cake scales approximately as

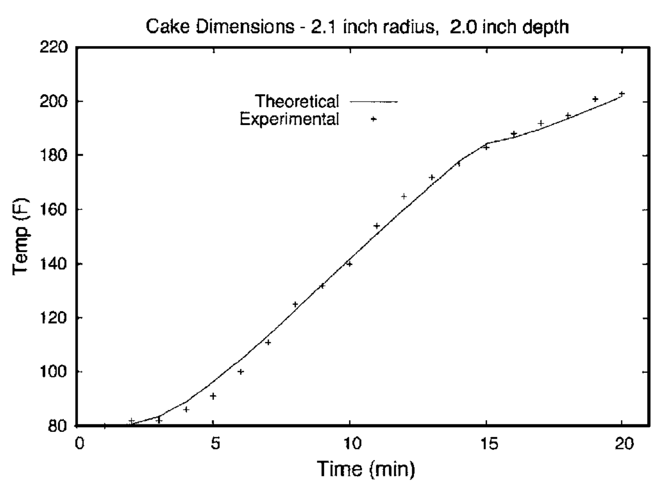

Fig. 5. A theoretical fit to the temperature as a function of time at the center of a cake, with a 2.1 in. radius and 2.0 in. depth. The parameters are given in Table II.

$$\frac{t}{Z^2}. \quad (17)$$

This type of behavior is typical in random walk processes because diffusion times scale as the square of some characteristic length.

Moisture also diffuses from the sides of the cake inward. If the radius of the cake is less than its depth, moisture in the interior of the cake might remain constant or possibly increase, even though moisture continues to evaporate from the top cake surface. For an amount of baking time t, we hypothesize that the moisture content scales as

$$\frac{t/R^2}{t/Z^2} = \frac{Z^2}{R^2}. \quad (18)$$

These hypotheses for the scaling behavior of the moisture content are highly speculative and the degree to which they can be justified rests on how successfully they can be applied to predicting the baking time of cakes of various dimensions.

IV. ESTIMATING THE BAKING TIME OF A CAKE

We now utilize the revised model of Sec. III B to obtain an explicit formula for the baking time of a cake.[26] In the revised model we have assumed that as a cake bakes its diffusivity changes, primarily because of evaporation of moisture at the top surface of the cake; however, for a given cake we can calculate using Eq. (5) an effective diffusivity $\mathcal{D}_{\text{eff}}$ that is constant during the baking process. In accord with the revised model the functional form of $\mathcal{D}_{\text{eff}}$ should be consistent with Eqs. (17) and (18). If the functional form is known, it provides a relation between the baking time and physical dimensions of the cake, making it possible to solve for the baking time as a function of the cake dimensions.

For each of the data entries in Table II we estimate an effective diffusivity using Eq. (5). We then fit these data to an expression for $\mathcal{D}_{\text{eff}}$ of the form,

$$\mathcal{D}_{\text{eff}} = a_0 - a_1 \exp\left[-a_2 \left(\frac{Z^2}{t_f} \right) \left(\frac{Z^2}{D^2} \right) \right]. \quad (19)$$

The parameters are estimated to be $a_0 = 2.57 \times 10^{-2}$, $a_1 = 2.07 \times 10^{-2}$, and $a_2 = 19.2$.

The functional form of Eq. (19) was chosen primarily because it can be interpreted in a relatively straightforward and physically intuitive manner. We have tried other forms with different dependencies on the baking time and cake dimensions; however, when inverted to estimate baking times they yield multiple roots for a cake of specified dimensions. Generally, these additional roots have no obvious physical interpretation. We have also considered functional forms that are not plagued by multiple roots for the baking time and have found the latter not to be too different from the results obtained using Eq. (19).

The effective heat diffusivity depends explicitly on the baking time, the diameter of the pan, and depth of the cake batter. In Figs. 6 and 7 we show the experimental values of $\mathcal{D}_{\text{eff}}$ and the theoretical fit. Rather than plotting $\mathcal{D}_{\text{eff}}$ as a function of the baking time, diameter of the pan, and depth of the batter, we have plotted $\mathcal{D}_{\text{eff}}$ as a function of t_f/Z^2 and D^2/Z^2.

The adjustable parameters in Eq. (19) have a simple physical interpretation. If the values of t_f, D, or Z are such that the exponential becomes vanishingly small, the cake batter should be maximally moist and $\mathcal{D}_{\text{eff}} = a_0$. If their values are such that the exponential approaches one, the cake should be totally dry with the effective heat diffusivity being $a_0 - a_1$. The parameter a_2 measures how readily moisture evaporates from the cake. The three parameters a_0, a_1, and a_2 should also depend on the initial moisture content of the cake batter.

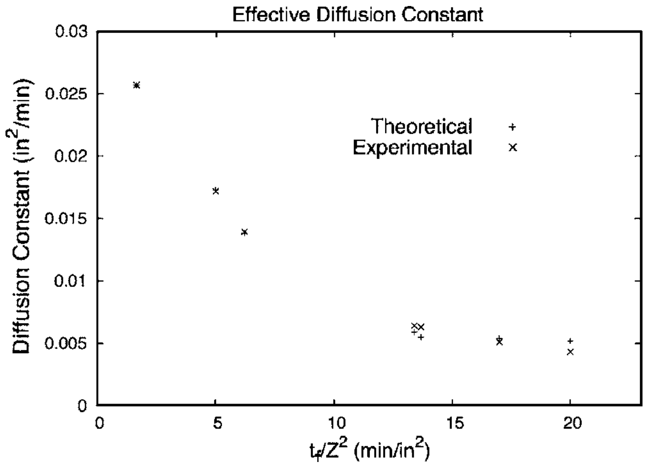

Fig. 6. The experimental values of the effective heat diffusivity and the theoretical fit to these values. The independent variable was selected based on the discussion of Sec. III B.

If we combine Eqs. (5) and (19) we obtain an implicit formula for the baking time as a function of depth, diameter, oven temperature, initial temperature of the cake batter, and final temperature of the cake:

$$K = -\mathcal{D}_{\text{eff}} t_f \begin{cases} \dfrac{4x_{1,0}^2}{D^2} + \dfrac{\pi^2}{Z^2} & (C_1 \leq 1 \text{ and } C_2 \leq 1) \\ \dfrac{\pi^2}{Z^2} & (C_1 > 1) \\ \dfrac{4x_{1,0}^2}{D^2} & (C_2 > 1), \end{cases} \quad (20)$$

where K is obtained from

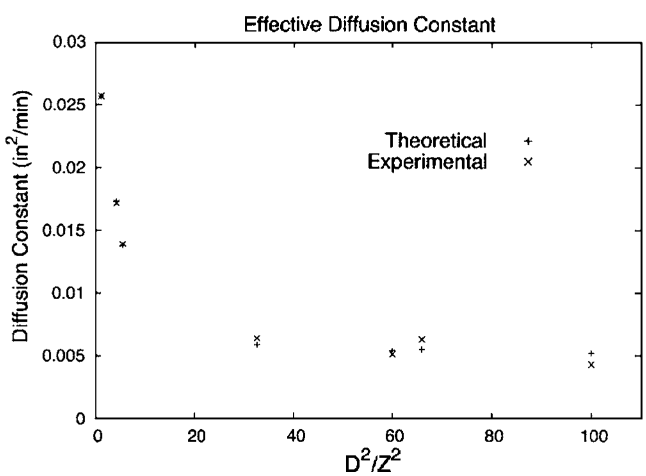

Fig. 7. The experimental values of the effective heat diffusivity and the theoretical fit to these values. The independent variable was selected based on the discussion of Sec. III B.

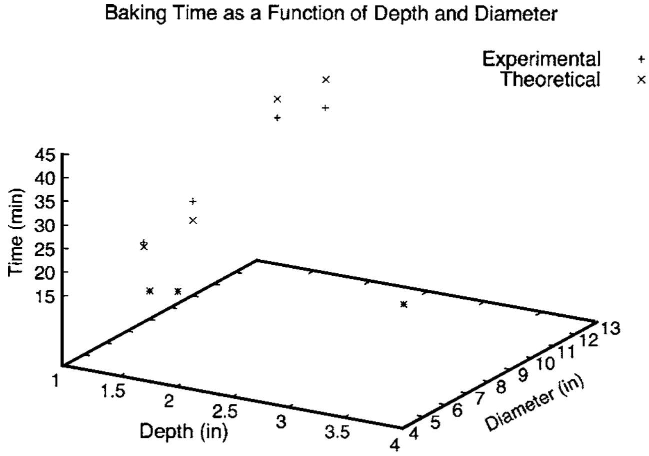

Fig. 8. The theoretical and measured baking times of cakes of various depths and diameters.

$$\exp(-K) = \left(\frac{T_f - T_i}{T_b - T_i}\right) \times \begin{cases} \dfrac{\pi x_{1,0} J_1(x_{1,0})}{8} & (C_1 \leqslant 1 \text{ and } C_2 \leqslant 1) \\ \dfrac{\pi}{4} & (C_1 > 1) \\ \dfrac{x_{1,0} J_1(x_{1,0})}{2} & (C_2 > 1). \end{cases} \quad (21)$$

The conditions C_1 and C_2 are defined by Eq. (7).

We use Eqs. (20) and (21) to estimate the baking times of cakes of various dimensions used in this study and compare them to the actual baking times. These results are presented in Fig. 8 and Table III. Note that Eqs. (20) and (21) exhibit the scaling behavior that is typical of diffusion processes, that is, the baking times are proportional to the square of a characteristic length.

Consider the four hypothetical cakes whose dimensions and predicted baking times are given in Table IV. For various reasons these cakes are impractical to bake. Nonetheless, they provide an arena for qualitatively exploring the theoretical basis of Eqs. (20) and (21). Even to someone with little cooking experience it is not surprising that cake two requires 17 min to bake, given that a cake 8 in. diameter and 1 in. depth requires 17 min baking time (see Table III). It may be surprising that cake three requires so little time to bake, given the baking time for cake two. The reason is that the moisture content of cake three remains high because very little evaporation takes place during the cooking process. As a result, the heat diffusivity remains relatively large so that the cake bakes quickly. Cake four requires nearly 24 h to bake. Note that the baking time of cake four reflects the scaling property of diffusion processes, that is, its physical dimensions are thirty times those of number one and is therefore predicted to require 900 times as long to bake.

Table III. Empirical baking times and times predicted by the theoretical model are presented for the diameters and the depths of cakes used in this study. The baking time predicted by linear regression is also presented.

t_f	t_{theor}	t_{linreg}	Diameter D	Depth Z
17	16	21	8.0	1.0
26	26	29	4.1	4.0
20	20	18	4.1	2.0
41	45	30	9.9	1.8
35	41	37	13.0	1.6
20	16	25	9.9	1.0
19	19	17	4.1	1.8

Table IV. The predicted baking times of four hypothetical cakes.

Cake	t_{theor}	Diameter D	Depth Z
1	1.56	1.00	1.00
2	17.0	30.0	1.00
3	1.82	1.00	30.0
4	1.40×10^3	30.0	30.0

The semiempirical model for the baking time has three adjustable parameters. We ask if performing a linear regression, which also has three adjustable parameters, would produce a model with as good or possibly better predictive ability than the revised model. The result of such an analysis for diameter and depth dependence is

$$t_f = -3.00 + 2.34D + 5.74Z. \quad (22)$$

In Table III we give the baking times predicted by Eq. (22) for comparison to those predicted according to theory. We argue as follows that the theoretical model is preferable to the linear regression. Using the data in Table III we calculate the sum of the squared residuals, that is, the differences between the predicted and actual value of baking time, for the linear regression and the theoretical model. For the regression model the sum is 183, while for the theoretical model the sum is 69. Each model has three adjustable parameters. Thus, from a statistical perspective the theoretical model is better than linear regression because there is less variance in the predicted baking time. The linear regression is not based on any physical principles underlying the baking process, thus making it less appealing than the theoretical model. In fact, the regression model predicts a negative baking time for a cake of zero thickness and diameter.

Although our analysis has been applied to the génoise, we can generalize the results. The difference between cake batter and pastry dough is the proportions of the same ingredients. Both typically are composed of flour, fats, and ingredients that contain water, eggs, milk, and juice. Typically, pastry dough contains a very small amount of liquid in proportion to other ingredients. The lack of liquid explains why it takes such a long time to pre-bake a pastry shell (15 to 30 min), even though pastry shells are generally not much more than $\frac{1}{8}$ to $\frac{1}{4}$ in. in depth. Cookie dough, which has a moisture content between pastry dough and cake batter, also requires a longer time to bake than cake batter. Cookies typically require between 10 and 15 min to bake, even though they are usually not more than $\frac{1}{2}$ in. thick.

V. CONCLUSIONS

The primary emphasis of this study has been to explain how modifying a cake recipe by changing either the dimensions of the cake or the amount of cake batter alters the baking time. Our analysis has been restricted to a particular type of cake, the génoise. We found that conduction is the primary mechanism of heat transfer and that the diffusion equation provides a theoretical framework for describing the baking process; however, the heat diffusivity does not remain constant during baking. The heat diffusivity changes during the baking process principally because of the evaporation of moisture at the top surface of the cake. We approximated the cake baking process as one in which the heat diffusivity assumes a constant value during baking. Its value depends on the diameter, depth, and baking time of the cake, all of which affect the moisture content of the cake. We then proposed a semiempirical formula for the effective heat diffusivity. The formula has three parameters whose values were estimated from the data. We inverted the formula to obtain the baking time of a cake as a function of diameter and depth. The resulting formula exhibits scaling behavior typical of diffusion processes: the baking time scales as the square of a characteristic size of the cake.

Although we have not solved the problem of how long is required to bake a cake, we have offered a qualitative explanation of the factors that most importantly determine baking times. In addition to the dimensions of the cake we suggest that the moisture content of a cake is the dominant factor affecting its baking time. This assumption explains why pastry dough and cookie dough require a relatively long time to bake in comparison to cake batter, after accounting for differences in their physical dimensions.

[a]Electronic mail: olszewski@uncw.edu

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[3]Robert L. Wolke, *What Einstein Told His Cook: Kitchen Science Explained* (Norton, New York, 2002).

[4]P. Barham, *The Science of Cooking* (Springer-Verlag, Berlin, 2001).

[5]B. Zanoni and C. Peri, "Study of the bread baking process," J. Food. Eng. **19**, 389–398 (1993).

[6]K. Thorvaldasson and C. Skjöldebrand, "Water diffusion in bread," Lebensm.-Wiss. Technol. **31**, 658–663 (1998).

[7]M. Lostie, R. Peczalski, J. Andrieu, and M. Laurent, "Study of sponge cake batter baking process. I. Experimental data," J. Food. Eng. **51**, 131–137 (2002).

[8]M. Lostie, R. Peczalski, J. Andrieu, and M. Laurent, "Study of sponge cake batter baking process. II. Modeling and parameter estimation," J. Food. Eng. **55**, 349–357 (2002).

[9]The baking process involves several stages before the cake is completely baked. The study in Refs. 7 and 8 applies only to the initial stage of the baking process.

[10]A comprehensive review of existing research is given in Ref. 8.

[11]The only significant modification to the recipe is that the eggs have not been warmed over boiling water prior to mixing.

[12]J. Pépin, *La Technique* (Times Books, New York, 1976), pp. 356–358.

[13]Reference 4, pp. 151–174, provides an excellent discussion of the physics and chemistry of baking sponge cakes, of which the génoise is an example.

[14]The cake exhibits a slight amount of shrinkage from the sides of the pan, and its top springs back, when lightly pressed.

[15]Klamkin has considered such a model for estimating the scaling behavior in the cooking time of a roast. See M. S. Klamkin, "On cooking a roast," SIAM Rev. **3**, 167–169 (1961).

[16]C. Kittel, *Thermal Physics* (Freeman, New York, 1980), pp. 424–425.

[17]George B. Arfken and Hans J. Weber, *Mathematical Methods for Physicists* (Academic, New York, 1995), 4th ed., pp. 473, 664.

[18]If the cake is rectangular, the term $2.34/D^2$ is replaced by $1/X^2 + 1/Y^2$, where X and Y are the length and width of the cake pan.

[19]The reader should be aware that some equations are expressed in terms of diameter rather than radius. The reason is to make transparent the relation between these formulas and those applicable to rectangular cake pans.

[20]Standard cake pans are not readily available with these dimensions. This lack has necessitated substituting a coffee can, filled with batter to a depth of 4.0 in. for a cake pan. In fact, génoises are commonly not baked greater than 2 in. in depth.

[21]Temperature display instruments are located outside of the oven, so that temperature data can be collected without opening the oven door.

[22]It has been necessary to truncate the sum at such large values of m and n ($m=17$, $n=17$) to obtain reasonable estimates of temperatures close to the initial temperature of the cake.

[23]We have observed that typically the mass of a baked cake is less by approximately 10%.

[24]Any time dependence of the thermal diffusivity is assumed to result from that of the thermal conductivity only. We neglect the possible time dependence of the mass density and specific heat. This neglect simplifies the analysis and avoids physically unrealistic, temporal discontinuities in the temperature which result from discontinuous changes in the thermal diffusivity.

[25]The computations were performed using MAPLE software. The optimization of adjustable parameters in the model was achieved by trial and error.

[26]In this section the diameter rather than the radius is used in all formulas.