Continuous-Time Models

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Continuous-Time Models with Differential Equations
Mathematical formulations of dynamical systems

• Discrete-time model: (difference/recurrence equations; iterative maps)
\[ x_t = F(x_{t-1}, t) \]

• Continuous-time model: (differential equations)
\[ \frac{dx}{dt} = F(x, t) \]

\( x_t \): State variable(s) of the system at time \( t \)
\( F \): Some function that determines the rule that the system’s behavior will obey

Including \( x \) in \( F( ) \) means “feedback loops”
A general form (first-order, autonomous)

\[
\begin{align*}
\frac{dx_1}{dt} &= F_1 (x_1, x_2, x_3, \ldots) \\
\frac{dx_2}{dt} &= F_2 (x_1, x_2, x_3, \ldots) \\
\frac{dx_3}{dt} &= F_3 (x_1, x_2, x_3, \ldots) \\
&\quad \ldots \\
\frac{dx}{dt} &= F(x)
\end{align*}
\]

or

\[
\frac{dx}{dt} = F(x)
\]

where \( x \) is a state vector of the system \( (x = \{x_1, x_2, x_3, \ldots\}) \)
Higher-order/non-autonomous systems

• Higher-order systems:
  Differential equations that include second-order (or higher) derivatives

• Non-autonomous systems:
  Differential equations that are time-dependent (i.e., explicitly include \( t \) in them)
The following argument also holds for differential equations

- Non-autonomous, higher-order equations can always be converted into autonomous, 1st-order equations
  \[ \frac{d^2x}{dt^2} \rightarrow \frac{dy}{dt}, \ y = \frac{dx}{dt} \]
  \[ -t \rightarrow y, \ \frac{dy}{dt} = 1, \ y_0 = 0 \]

- Autonomous 1st-order equations can cover dynamics of any non-autonomous higher-order equations too!
Connecting continuous-time models with discrete-time models

\[ x_t = F(x_{t-1}) \quad \text{dx}/\text{dt} = G(x) \]

- \( F(x) \iff x + G(x) \Delta t \)
- \( G(x) \iff (F(x) - x) / \Delta t \)

- If \( F(x) = Ax, \ G(x) = Bx \):
  - \( A \iff I + B \Delta t \)
  - \( B \iff (A - I) / \Delta t \)
How to study differential equations

• Some of them can be analytically solvable
  - Linear systems
  - Simple nonlinear systems

• Analytical solutions are generally not available for nonlinear differential equations
Numerical simulation

• Simplest way: Euler forward method

\[ \frac{dx}{dt} = F(x) \]

\[ \rightarrow x_{t+\Delta t} = x_t + F(x) \Delta t \]

• Approximate dynamics using small discrete time steps (\( \Delta t \ll 1 \))

• Simulate the model like difference equations
Exercise

• Simulate the following continuous-time logistic growth model in Python, with $r=0.2$, $K=1$, $\Delta t=0.01$:

$$\frac{dN}{dt} = r \cdot N \left( 1 - \frac{N}{K} \right)$$
Analysis of Continuous-Time Models
Equilibrium point

- A state of the system at which state will not change over time
  - A.k.a. fixed point, steady state
- Can be calculated by solving

\[ \frac{dx}{dt} = 0 \]
Example

- A simple second-order equation:
  \[ \frac{d^2x}{dt^2} = x \]

- Convert this into a first-order form
- Calculate its equilibrium points
Exercise

• A simple pendulum:

\[ \frac{d^2\theta}{dt^2} = -\frac{g}{L} \sin \theta \]

• Convert this into a first-order form
• Calculate its equilibrium points
Visualizing Phase Space of Continuous Models
Phase space of continuous models

- E.g. a simple vertical spring oscillator
- State can be specified by two real variables (location $x$, velocity $v$)

Dynamics of continuous models can be depicted as "flow" in a continuous phase space.
Visualizing phase space of continuous models

- **Vector field**
  - Uses many small arrows to show how local derivatives (or direction of trajectories) change from place to place in phase space.

- **Phase portrait (stream plot)**
  - Shows several typical trajectories to illustrate how phase space is globally structured.
Exercise

• Write your own code to visualize the phase space of the simple pendulum model

\[ \frac{d^2\theta}{dt^2} = - \frac{g}{L} \sin \theta \]

• Include some “damping” effect in the model above and see how it changes the phase space
Visualizing phase space of continuous models manually

- Find “nullclines”
  - Points in the phase space where one of the derivatives is zero (i.e., trajectories are in parallel to one of the axes)
  - Plot where the nullclines are
  - Find how the sign of the derivative changes across the nullclines
  - Find values of other non-zero derivatives

- Draw a “flow” between those nullclines with curves that don’t intersect with each other
Exercise

• Draw an outline of the phase space of the following system by studying the distribution of its nullclines:

\[
\begin{align*}
\frac{dx}{dt} &= ax - bx \ y \\
\frac{dy}{dt} &= -cy + dx \ y
\end{align*}
\]

\((x \geq 0, \ y \geq 0)\)
Rescaling Variables
Rescaling variables

• Dynamics of a system won’t change qualitatively by linear rescaling of variables (e.g., \( x \rightarrow \alpha x' \))

• You can set arbitrary rescaling factors for variables to simplify the model equations

• If you have \( k \) variables (including \( t \)), you may eliminate \( k \) parameters
Exercise

• Simplify the logistic growth model by rescaling $N \rightarrow \alpha N'$ and $t \rightarrow \beta t'$

\[
dN/dt = r N (1 - N/K)
\]
Asymptotic Behavior of Linear Systems
Linear systems

- Linear systems are the simplest cases where states of nodes are continuous-valued and their dynamics are described by a time-invariant matrix

- Continuous-time: $\frac{dx}{dt} = A x$
  - $A$ is called a “coefficient” matrix
  - We don’t consider constants (as they can be easily converted to the above forms)
Where will the system go eventually?

\[ \frac{dx}{dt} = A \cdot x \]

These equations give the following exact solution:

\[ x_t = e^{A t} \cdot x_0 \]

\[ = \sum_{k=0}^{\infty} \frac{(A t)^k}{k!} \cdot x_0 \]
FYI: Exponential operator for matrices

\[ x_t = e^{At} x_0 = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} x_0 \]

- Similar to the Taylor series expansion of the exponential function:
  \[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \]
- \( e^M \) converges for any square matrix \( M \)
- If \( M \)'s eigenvalues are \( \{\lambda_i\} \), then \( e^M \)'s eigenvalues are \( \{ e^{\lambda_i} \} \), with all eigenvectors unchanged (you can prove this)
Where will the system go eventually?

\[ \frac{dx}{dt} = Ax \]

- What happens if the system starts from non-equilibrium initial states and goes on for a long period of time?

- Let's think about their asymptotic behavior \( \lim_{t \to \infty} x(t) \)
Considering asymptotic behavior (1)

- Let \{ v_i \} be \textit{n linearly independent eigenvectors} of the coefficient matrix (They might be fewer than \textit{n}, but here we ignore such cases for simplicity)

- Write the initial condition using eigenvectors, i.e.
  \[
  x_0 = b_1v_1 + b_2v_2 + \ldots + b_nv_n
  \]
Considering asymptotic behavior (2)

• Then:

\[ x_t = e^{A t} \cdot x_0 \]

\[ = e^{\lambda_1 t} \cdot b_1 v_1 + e^{\lambda_2 t} \cdot b_2 v_2 + \ldots + e^{\lambda_n t} \cdot b_n v_n \]
Dominant eigenvector

- If $\text{Re}(\lambda_1) > \text{Re}(\lambda_2), \text{Re}(\lambda_3), \ldots$,

$\mathbf{x}_t = e^{\lambda_1 t} \{ b_1 \mathbf{v}_1 + e^{(\lambda_2-\lambda_1)t} b_2 \mathbf{v}_2 + \ldots + e^{(\lambda_n-\lambda_1)t} b_n \mathbf{v}_n \}$

$\lim_{t \to \infty} \mathbf{x}_t \sim e^{\lambda_1 t} b_1 \mathbf{v}_1$

If the system has just one such dominant eigenvector $\mathbf{v}_1$, its state will be eventually along that vector regardless of where it starts
What eigenvalues and eigenvectors can tell us

- An eigenvalue tells whether a particular “state” of the system (specified by its corresponding eigenvectors) grows or shrinks by interactions between parts
  - $|\lambda| > 1 \rightarrow$ growing
  - $|\lambda| < 1 \rightarrow$ shrinking
  - $\text{Re}(\lambda) > 0 \rightarrow$ growing
  - $\text{Re}(\lambda) < 0 \rightarrow$ shrinking

for discrete-time cases

for continuous-time cases
Linear Stability Analysis of Nonlinear Systems
Linearizing continuous-time models

- For continuous-time models:
  \[ \frac{dx}{dt} = F(x) \]
  
  \[ \text{Left} = \frac{d(x_e + \Delta x)}{dt} = \frac{d\Delta x}{dt} \]
  
  \[ \text{Right} = F(x_e + \Delta x) \]
  
  \[ \sim F(x_e) + F'(x_e) \Delta x \]
  
  \[ = F'(x_e) \Delta x \]

  Therefore,
  \[ \frac{d\Delta x}{dt} = F'(x_e) \Delta x \]
Review: First-order derivative of vector functions

- **Continuous-time:** $\frac{d\Delta x}{dt} = F'(x_e) \Delta x$

These can hold even if $x$ is a vector

What corresponds to the first-order derivative in such a case:

$$F'(x_e) = \frac{dF}{dx}(x=x_e) = \begin{pmatrix}
\frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\
\frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_n}
\end{pmatrix}(x=x_e)$$

Jacobian matrix at $x=x_e$
Eigenvalues of Jacobian matrix

- A Jacobian matrix is a linear approximation around the equilibrium point, telling you the local dynamics: “how a small perturbation will grow, shrink or rotate around that point”
  - The equilibrium point serves as a local origin
  - The $\Delta x$ serves as a local coordinate
  - Eigenvalue analysis applies