Ch1_Calc I

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1.1 Slope of a Curve

Straight Lines

$$y = mx + b$$

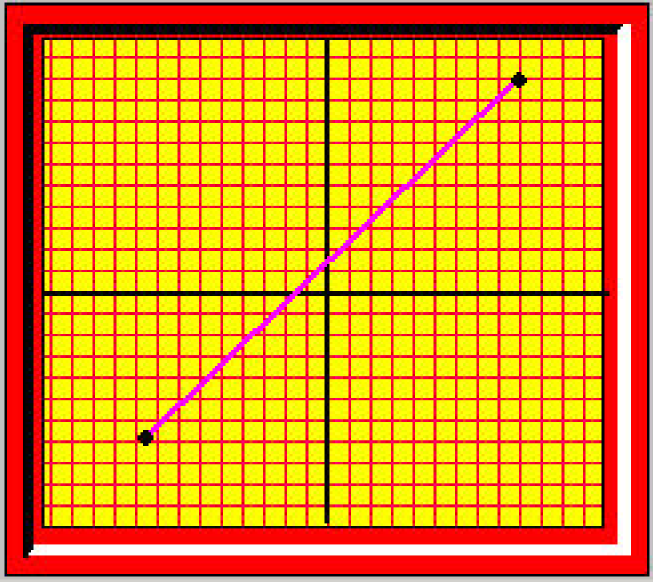
1) Main Facts

m = slope of the line

- $m > 0$. Line going uphill
- $m < 0$. Line going downhill
- $m = 0$. Line is horizontal

Graphs

m > 0	increasing



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1.1 Slope of a Curve

Slope of a Curve

$$y = f(x)$$

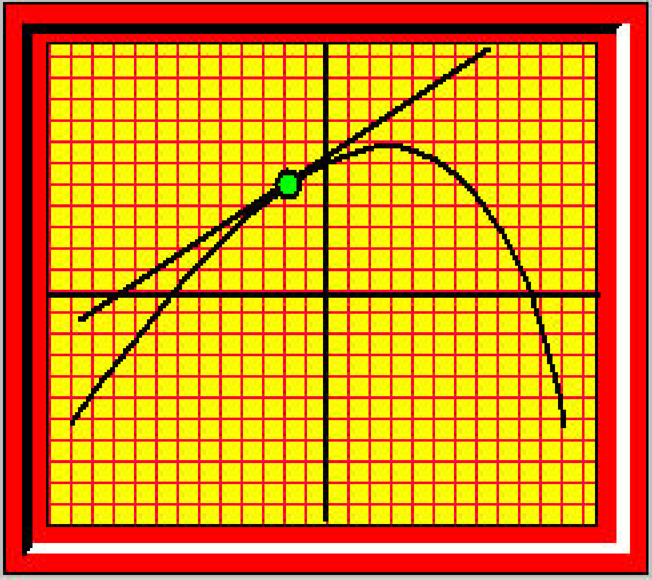
Definition. We define the slope of the curve to be the slope m of the tangent to $f(x)$.

$m > 0$. Curve going uphill

$m = 0$. Curve is horizontal

$m < 0$. Curve going downhill

Graphs



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Computation of Limits

EXAMPLE

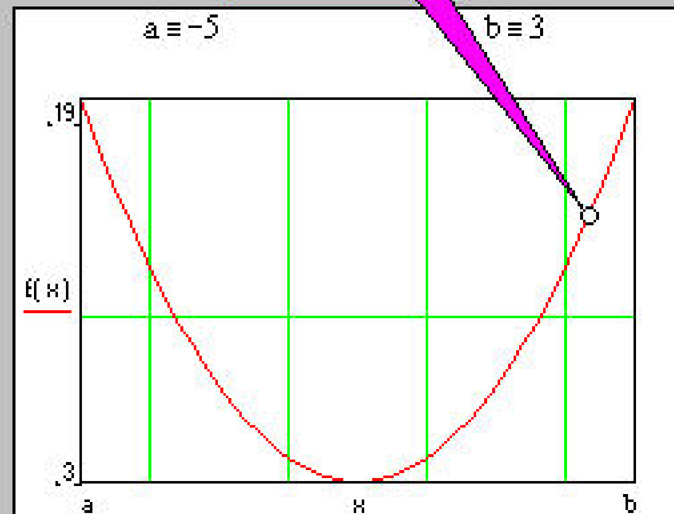
$$\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$$

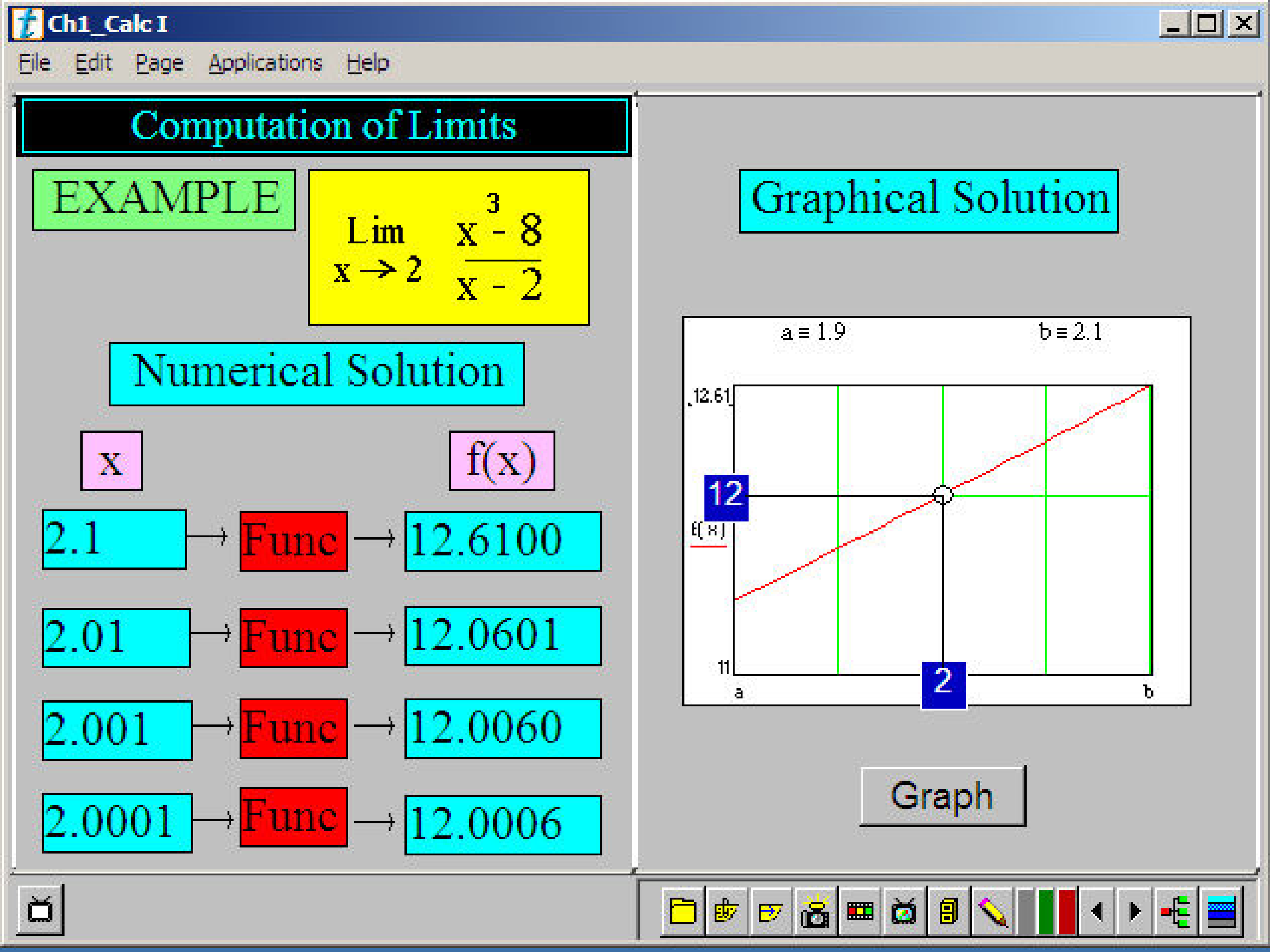
Numerical Solution

x		f(x)
2.1	→ Func →	12.6100
2.01	→ Func →	12.0601
2.001	→ Func →	12.0060
2.0001	→ Func →	12.0006

Graphical Solution

ZOOM

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Computation of Limits
EXAMPLE
$\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$
Numerical Solution
x
f(x)
2.1 → Func → 12.6100
2.01 → Func → 12.0601
2.001 → Func → 12.0060
2.0001 → Func → 12.0006
Graphical Solution
a = 1.9
b = 2.1
12.61
12
f(x)
11
a
2
b
Graph

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Computation of Limits

EXAMPLE

Analytic Solution

$$\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$$

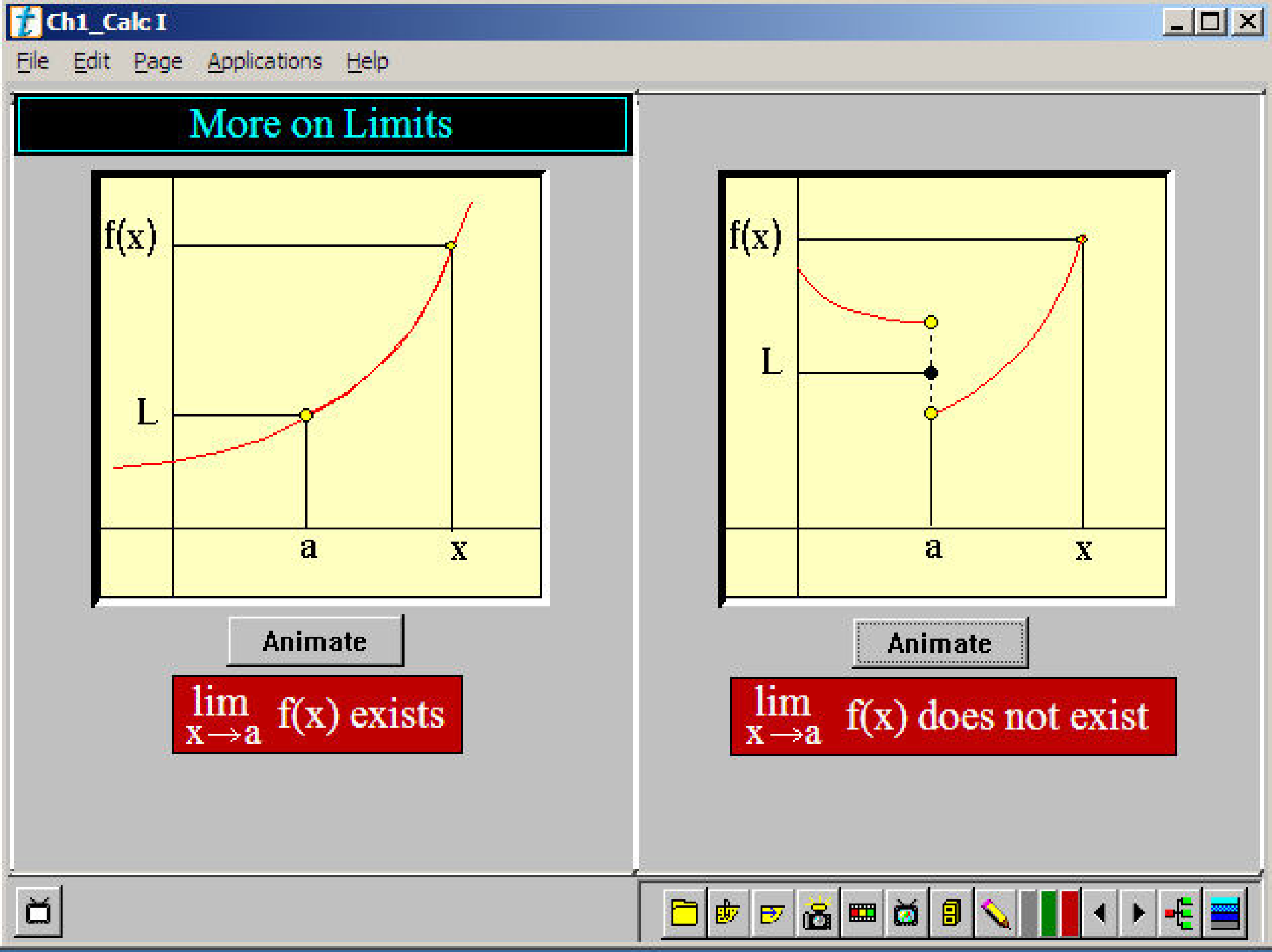
$$\lim_{x \to 2} \frac{x^3 - 8}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x^2 + 2x + 4)}{(x-2)}$$

$$= \lim_{x \to 2} (x^2 + 2x + 4)$$

$$= (2^2 + 2(2) + 4)$$

$$\lim_{x \to 2} \frac{x^3 - 8}{x - 2} = 12$$

Continue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More on Limits
f(x)
L
a
x
Animate
$\lim_{x \to a} f(x)$ exists
f(x)
L
a
x
Animate
$\lim_{x \to a} f(x)$ does not exist

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One-Sided Limits

Example

Problem 1

$$f(x) = \begin{cases} 2x + 1 & \text{if } x < 1 \\ 2 - x & \text{if } x > 1 \end{cases}$$

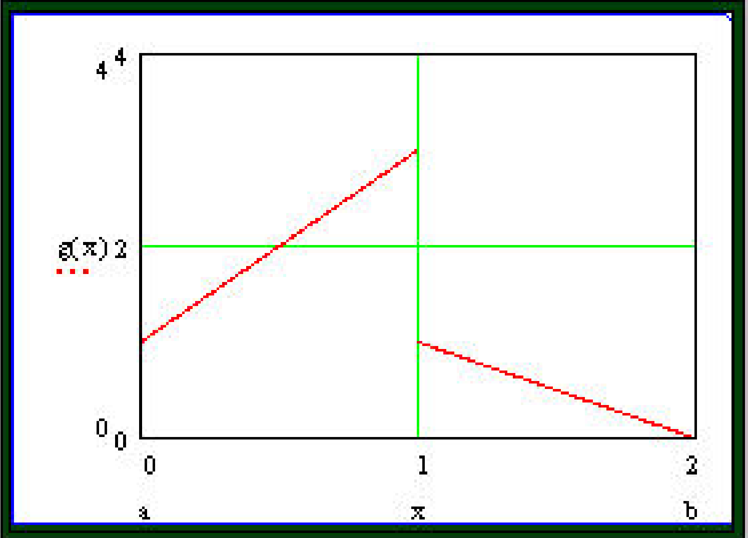
Find: $\lim_{x \to 1^+} f(x)$

Solution 1

Graph

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} (2 - x)$$

$$= 1$$



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One-Sided Limits

Example

Problem 2

$$f(x) = \begin{cases} 2x + 1 & \text{if } x < 1 \\ 2 - x & \text{if } x > 1 \end{cases}$$

Find: $\lim\limits_{x \to 1^-} f(x)$

Solution 2

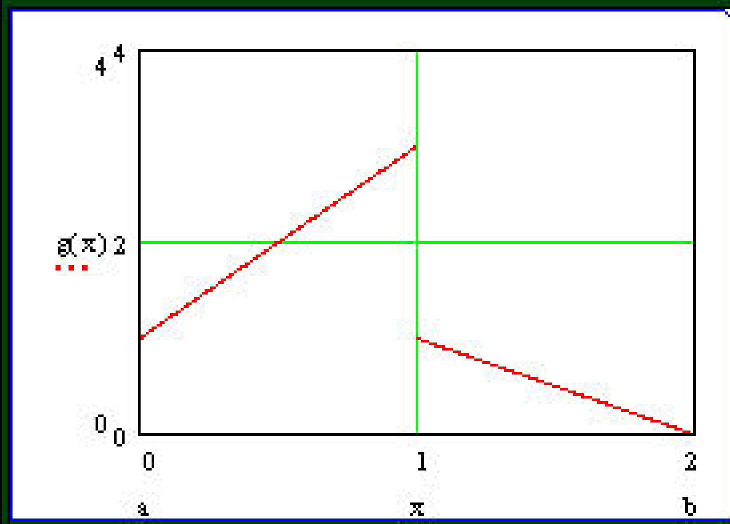
Graph

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} (2 - x)$$

$$= 1$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} (2x + 1)$$

$$= 3$$



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One-Sided Limits

Example

Problem 3

$$f(x) = \begin{cases} 2x + 1 & \text{if } x < 1 \\ 2 - x & \text{if } x > 1 \end{cases}$$

Find: $\lim_{x \to 1} f(x)$

Solution 3

Graph

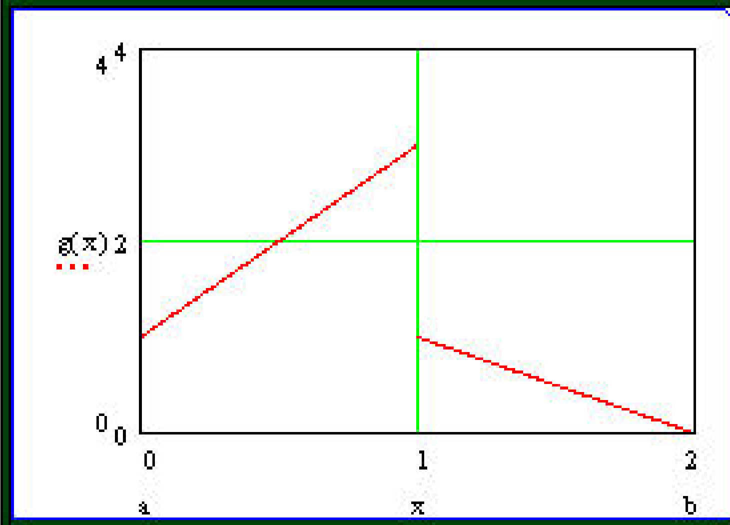
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1} (2 - x)$$

$$= 1$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1} (2x + 1)$$

$$= 3$$

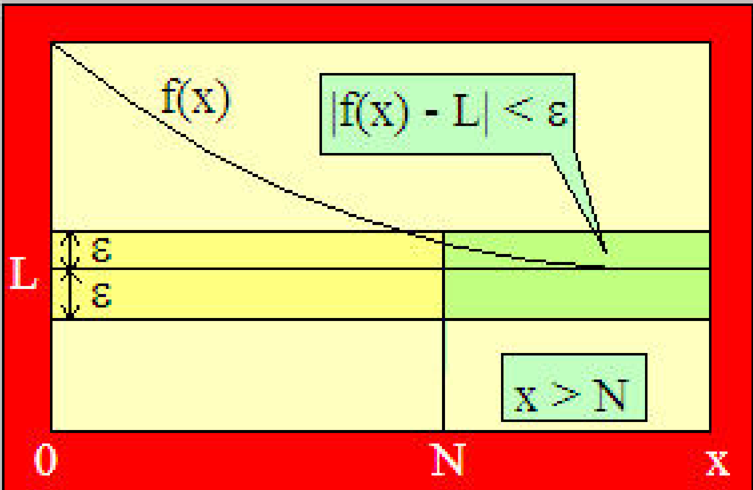
The limit of the given function as x goes to 1 does not exist.



1.4 Limits Involving Infinity.

Definition

The limit as x goes to infinity of a function f(x) is L if, for any $\varepsilon > 0$, there exists a number $N > 0$, such that, $|f(x) - L| < \varepsilon$, whenever $x > N$.



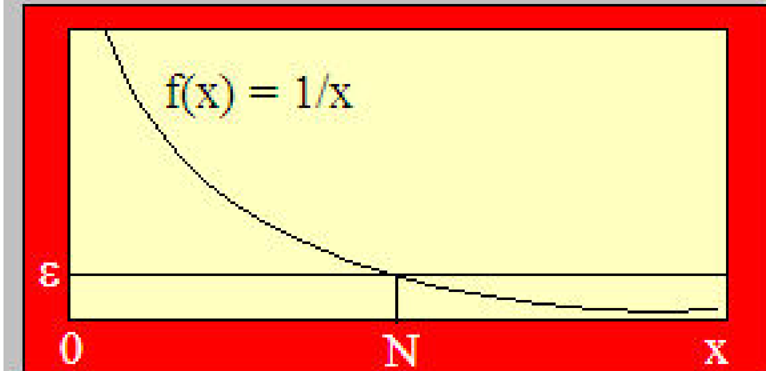
Example

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Proof

Given any number $\varepsilon > 0$, choose $N = 1/\varepsilon$. Then, whenever $x > N$,

$$|f(x) - L| = |(1/x) - 0|$$
$$= |1/x| < |1/N|$$
$$< \varepsilon$$



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Limit Theorems

Theorem. If $f(x) \to L$, and

$g(x) \to M$,

as x approaches to a, then,

1) $cf(x) \to cL$
2) $f(x) + g(x) \to L + M$
3) $f(x) - g(x) \to L - M$
4) $f(x) \cdot g(x) \to L \cdot M$
5) $f(x) / g(x) \to L / M$, if $M \neq 0$

Example

1) $\lim_{x\to 5} x^2 = \lim_{x\to 5} x \cdot \lim_{x\to 5} x$

$= 5 \cdot 5 = 25$

2) $\lim_{x\to 3} x^4 = \left[\lim_{x\to 3} x\right]^4 = 3^4 = 81$

Theorem. If P (x) is a polynomial of deg n, then: $\lim_{x\to a} P_n(x) = P_n(a)$

Example:

3) $\lim_{x\to 3} x^4 - 3x^2 + 1 = 3^4 - 3 \cdot 3^2 + 1$

$= 81 - 27 + 1$

$= 55$

Theorem. If the limit as $x \to a$ of a function f(x) exists, then it is

Example: Discuss

4) $\lim_{x\to 0} \sin(1/x)$

5) $\lim_{x\to 0} x \sin(1/x)$

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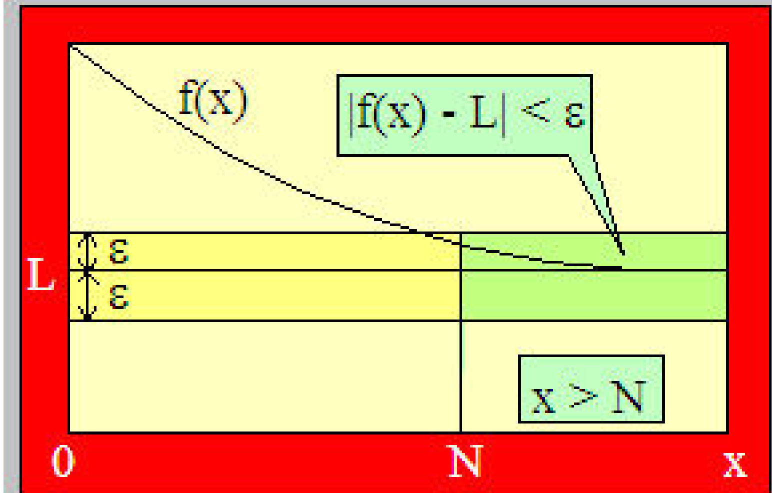
1.4 Limits Involving Infinity.

Definition

The limit as x goes to infinity of a function f(x) is L if, for any $\varepsilon > 0$, there exists a number $N > 0$, such that, $|f(x) - L| < \varepsilon$, whenever $x > N$.



Theorem. If $g(x) \le f(x) \le h(x)$ and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$, (Sandwich Thm.)

Ex.

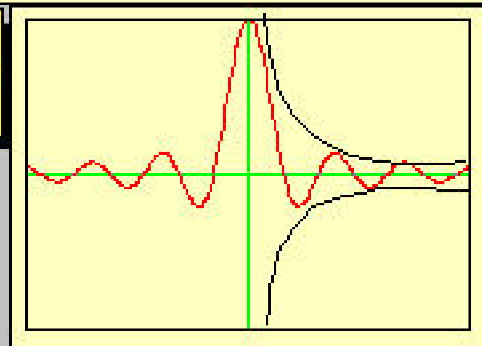
$$-\frac{1}{x} < \frac{\sin x}{x} < \frac{1}{x}$$

$$\lim_{x\to\infty} -\frac{1}{x} \le \lim_{x\to\infty} \frac{\sin x}{x} \le \lim_{x\to\infty} \frac{1}{x}$$

$$0 \le \lim_{x\to\infty} \frac{\sin x}{x} \le 0$$

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0$$

Graph



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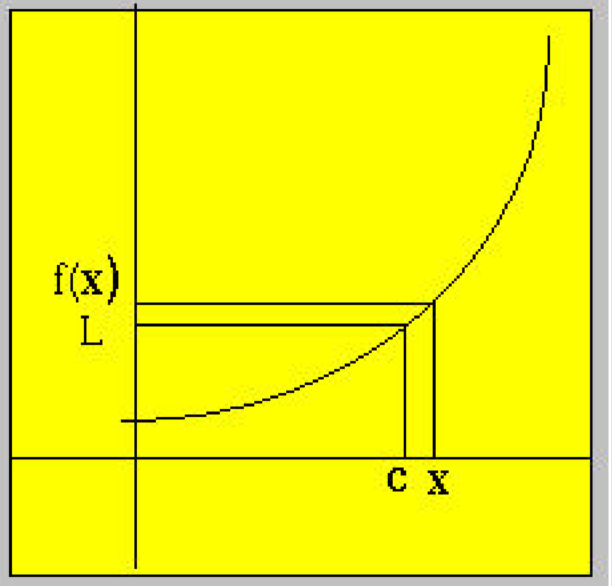
Definition of Limits

Limits

$$\lim_{x \to c} f(x) = L$$

1) INTUITIVE DEFINITION

The limit as x goes to c of f(x) is L if f(x) is close to L, whenever x is close to c



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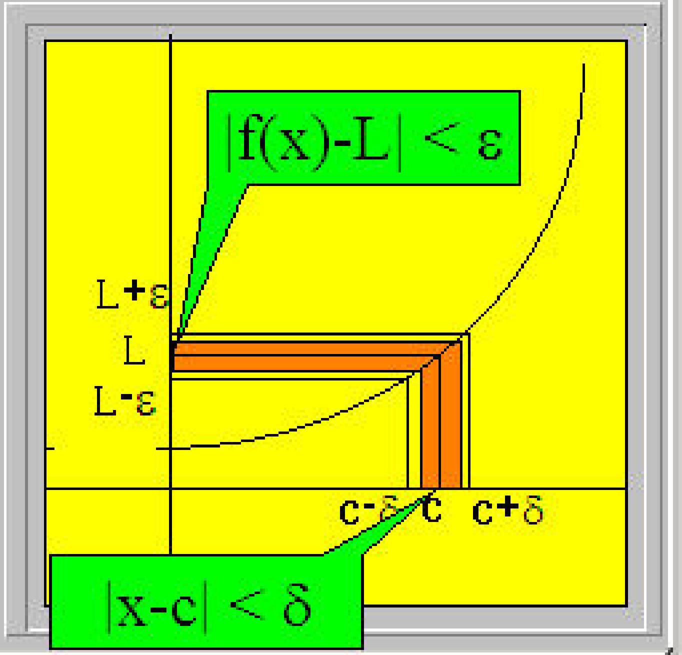
Definition of Limits

Limits

$$\lim_{x \to c} f(x) = L$$

1) RIGOROUS DEFINITION

The limit as x approaches c of a funtion f(x) is L, if for any number $\varepsilon > 0$, there exists a number $\delta > 0$, such that $|f(x)-L|<\varepsilon$, whenever $|x - c| < \delta$.



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Continuity

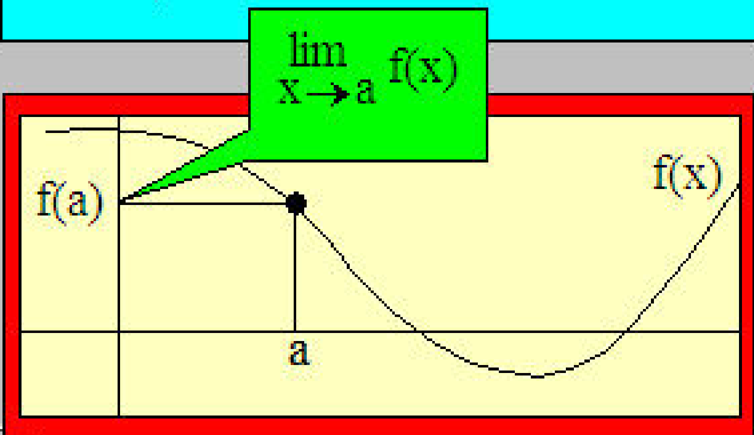
Definition

A function **f(x) is continuous at a** point $x = a$, if:

1) $f(a)$ is defined

2) $\lim_{x \to a} f(x)$ exists

3) $\lim_{x \to a} f(x) = f(a)$

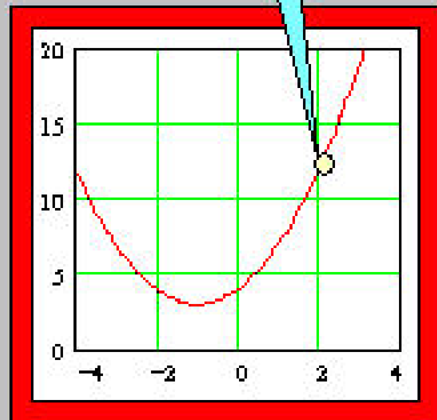


Ex. 1) Is f(x) cont. at x = 2?

$$f(x) = \frac{x^3 - 8}{x - 2}$$

Solution

No, because $f(2)$ is not defined.

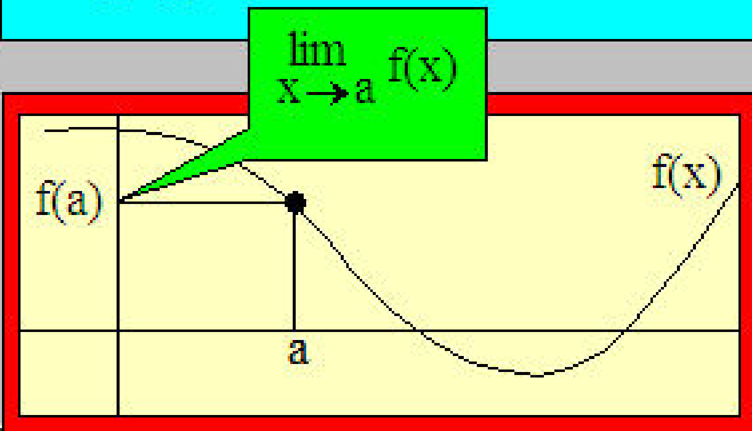


Continuity

Definition

A function f(x) is **continuous** at a point $x = a$, if:

1) f(a) is defined
2) $\lim_{x \to a} f(x)$ exists
3) $\lim_{x \to a} f(x) = f(a)$



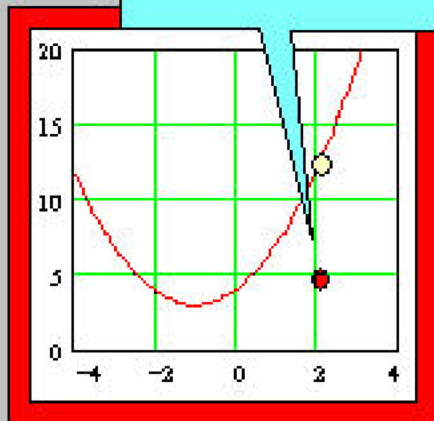
Ex. 2)

Is f(x) cont. at $x = 2$?

$$f(x) = \begin{cases} \dfrac{x^3 - 8}{x - 2} & \text{if } x \neq 2 \\ 5 & \text{if } x = 2 \end{cases}$$

Solution

No. $\lim_{x \to 2} f(x) \neq f(2)$



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Continuity

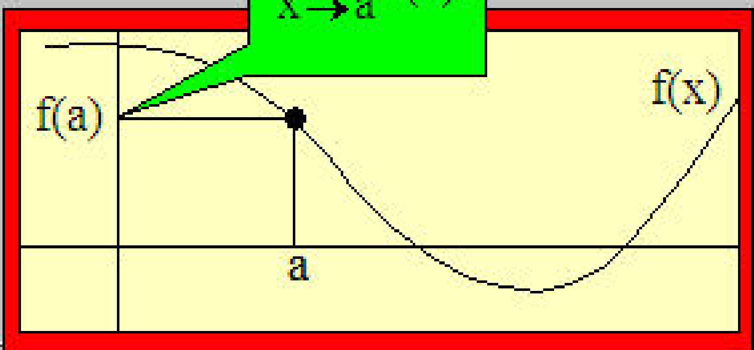
Definition

A function f(x) is **continuous** at a point x = a, if:

1) f(a) is defined

2) $\lim_{x \to a} f(x)$ exists

3) $\lim_{x \to a} f(x) = f(a)$

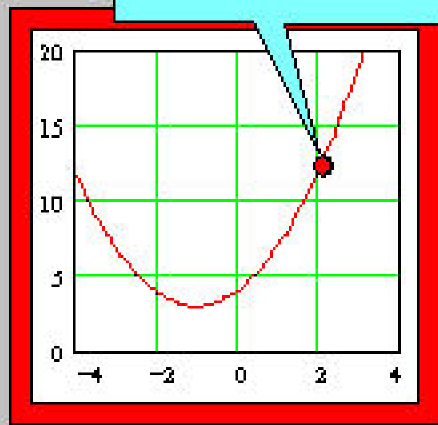


Ex. 3) Is f(x) cont. at x = 2?

$$f(x) = \begin{cases} \dfrac{x^3 - 8}{x - 2} & \text{if } x \neq 2 \\ 12 & \text{if } x = 2 \end{cases}$$

Solution

Yes. $\lim_{x \to 2} f(x) = f(2)$



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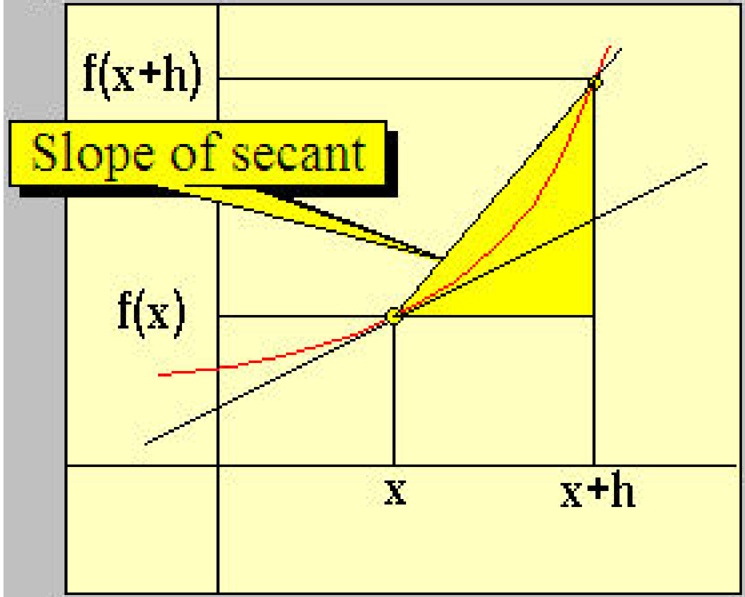
2.1 Derivatives

Definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



Notation $y = f(x) \quad h = \Delta x$

$$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

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Differentiation Rules

$$y = f(u),\ \ u = g(x),\ v = h(x)$$

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

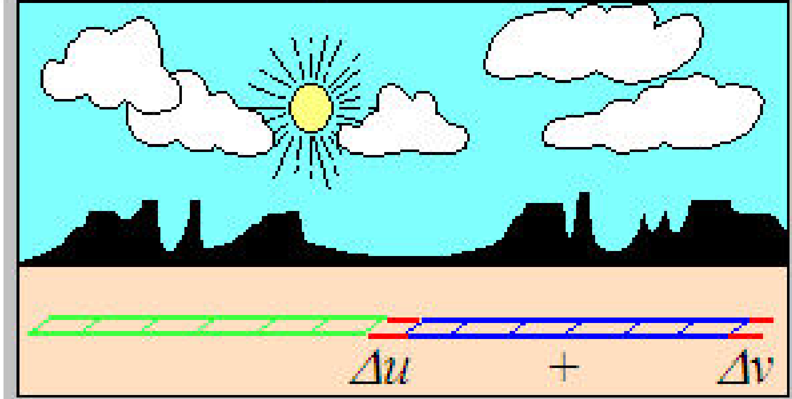
Ex1

$$\frac{d}{dx}(u\,v) = u\,\frac{dv}{dx} + v\,\frac{du}{dx}$$

$$\frac{d}{dx}\,\frac{u}{v} = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

Sum Rule



$$\Delta(u+v) = \Delta u + \Delta v$$

$$\frac{\Delta(u+v)}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$$

Taking the limit as Δx goes to 0

$$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

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Differentiation Rules

$y = f(u), \; u = g(x), \; v = h(x)$

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

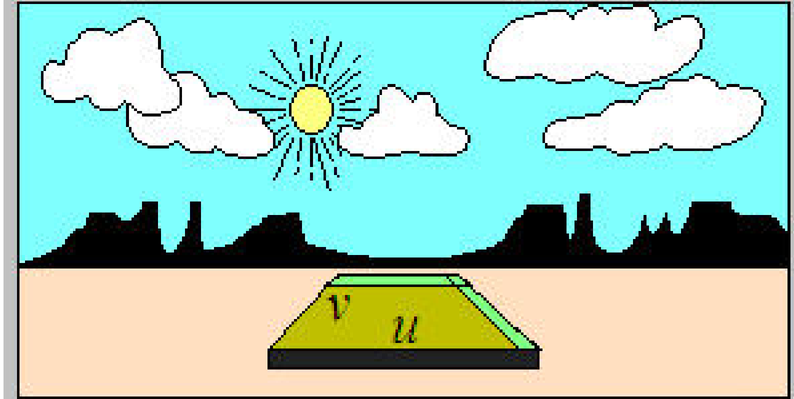
$$\frac{d}{dx}(u\,v) = u\,\frac{dv}{dx} + v\,\frac{du}{dx}$$

Ex2

$$\frac{d}{dx}\,\frac{u}{v} = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Product Rule

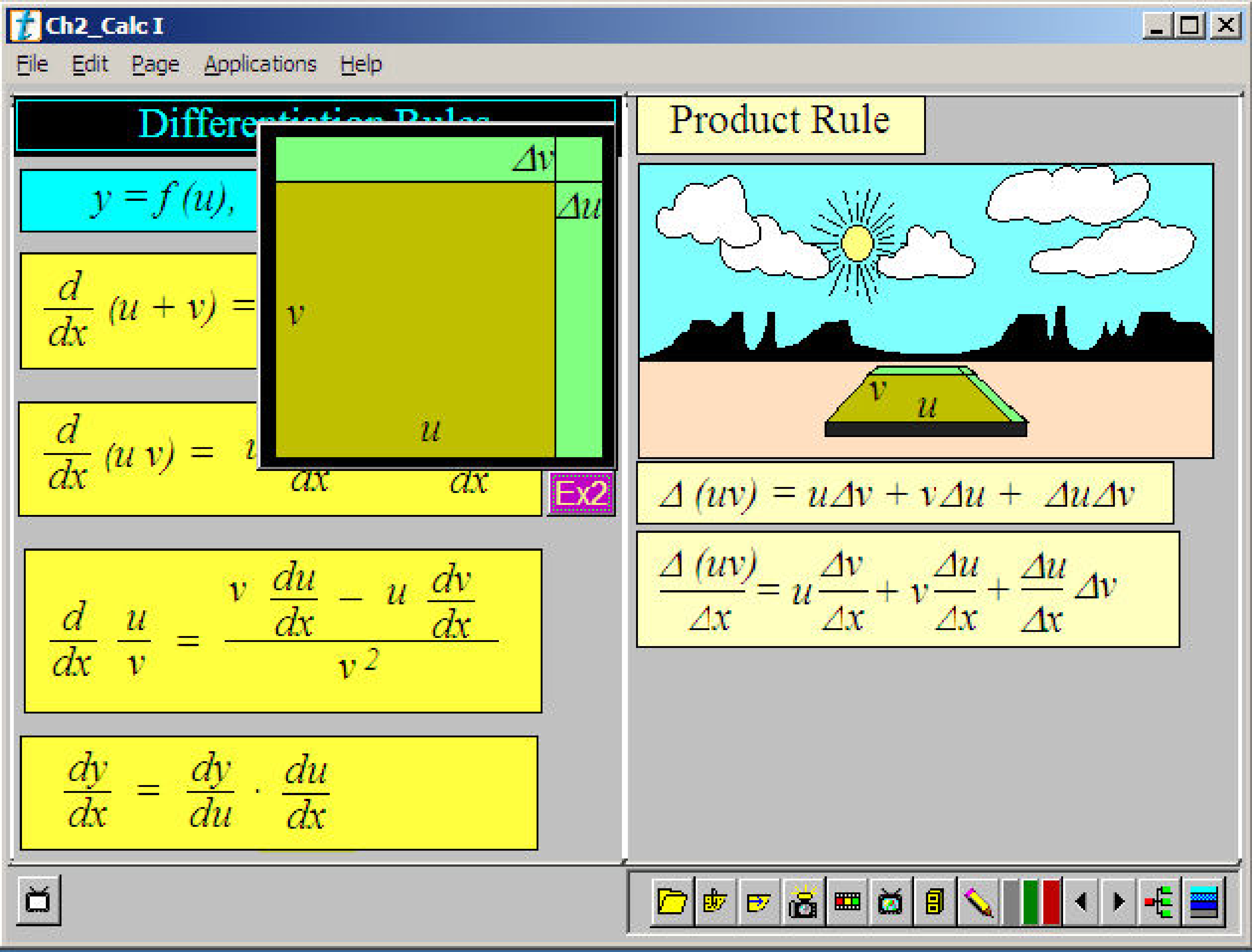


$$\Delta(uv) = u\Delta v + v\Delta u + \Delta u\Delta v$$

$$\frac{\Delta(uv)}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x}\Delta v$$

Taking the limit as Δx goes

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$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Differentiation Rules
y = f (u),
d/dx (u + v) =
d/dx (u v) =
d/dx (u/v) = (v du/dx − u dv/dx) / v²
dy/dx = dy/du · du/dx
Δv
Δu
v
u
Ex2
Product Rule
v
u
Δ (uv) = uΔv + vΔu + ΔuΔv
Δ (uv)/Δx = u Δv/Δx + v Δu/Δx + Δu/Δx Δv

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Differentiation Rules

$$y = f(u),\ u=g(x),\ v=h(x)$$

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

$$\frac{d}{dx}(u\,v) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Ex3

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

Quotient Rule

$$\frac{d\,(zv)}{dx} = z\frac{dv}{dx} + v\frac{dz}{dx}$$

$$v\frac{dz}{dx} = \frac{d(zv)}{dx} - z\frac{dv}{dx}$$

Let $z = \frac{u}{v}$

$$v\frac{d}{dx}\frac{u}{v} = \frac{du}{dx} - \frac{u}{v}\frac{dv}{dx}$$

$$v\frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v}$$

$$\frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

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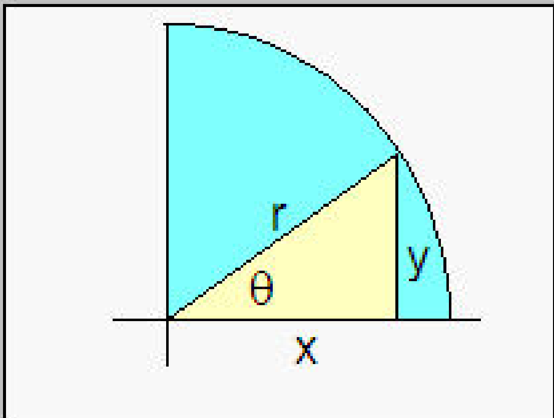
Trigonometric Functions (1)

Brief Review



$\sin\theta = y/r$

$\csc\theta = r/y$

$\cos\theta = x/r$

$\sec\theta = r/x$

$\tan\theta = y/x$

$\cot\theta = x/y$

$x = r\cos\theta$

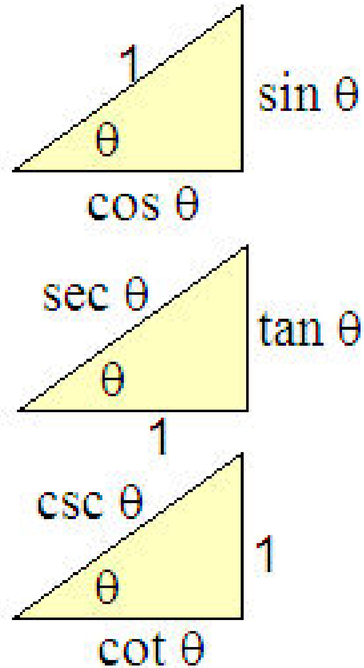
$y = r\sin\theta$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$



$\sin 0 = 0$

$\cos 0 = 1$

$\sin \pi/2 = 1$

$\cos \pi/2 = 0$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$\cot^2\theta + 1 = \csc^2\theta$

Cont

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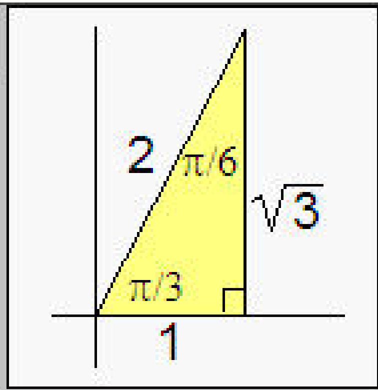
Trigonometric Functions (2)

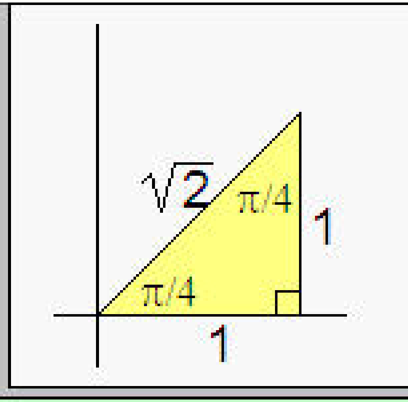
Sum formulas

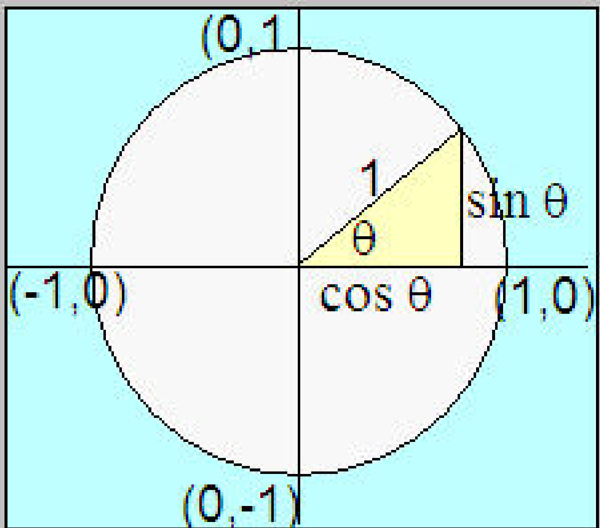
$\sin(\theta+\phi)=\sin\theta\cos\phi+\sin\phi\cos\theta$

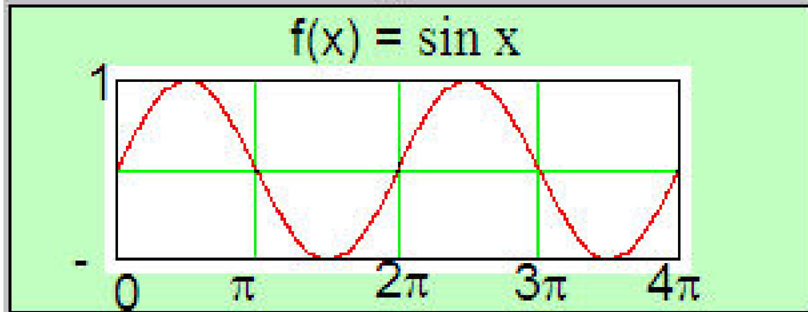
$\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\phi\sin\theta$

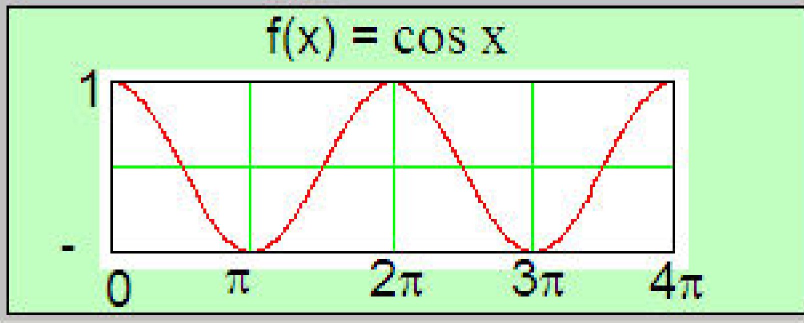
Special Angles











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Derivatives of Trig Functions

Derivative of f(x)=sin x

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\sin x\cos h + \sin h\cos x - \sin x}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \sin h\cos x}{h}$$

$$\frac{dy}{dx} = \lim_{h\to 0} \sin x\frac{(\cos h - 1)}{h} + \lim_{h\to 0} \cos x\frac{\sin h}{h}$$

$$\frac{dy}{dx} = \sin x \lim_{h\to 0} \frac{(\cos h - 1)}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h}$$

$$\frac{dy}{dx} = \sin x \cdot 0 + \cos x \cdot 1$$

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Derivatives of Trig Functions

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

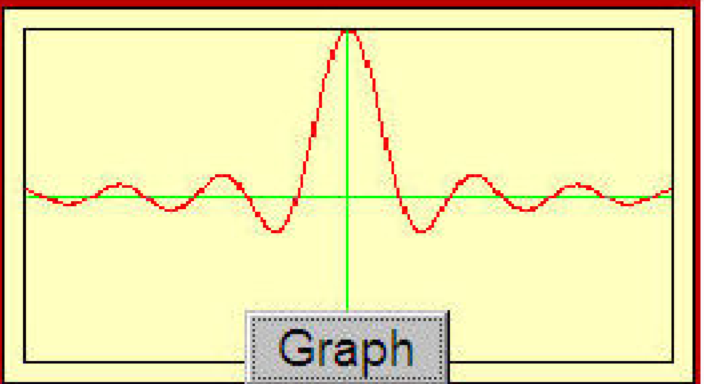
$$\frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\csc x = -\csc x \cot x$$

Computation of Limits

Discussion

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$



Ex.

$$\lim_{x \to 0} \frac{\sin 4x}{3x} = \frac{4}{3} \lim_{x \to 0} \frac{\sin 4x}{4x}$$

$$= \frac{4}{3} \lim_{\theta \to 0} \frac{\sin \theta}{\theta}$$

$$= 4/3$$

Example

$$\lim_{x \to 0} \frac{\tan^2 3x}{5x^2} = \frac{9}{5} \lim_{x \to 0} \frac{\tan^2 3x}{9x^2}$$

$$= \frac{9}{5} \lim_{x \to 0} \frac{\sin^2 3x}{9x^2 \cos^2 3x}$$

$$= \frac{9}{5} \lim_{x \to 0} \left[\frac{\sin 3x}{3x}\right]^2 \frac{1}{\cos^2 3x}$$

$$= \frac{9}{5} \lim_{x \to 0} \left[\frac{\sin 3x}{3x}\right]^2 \cdot \lim_{x \to 0} \frac{1}{\cos^2 3x}$$

$$= \frac{9}{5} \lim_{\theta \to 0} \left[\frac{\sin \theta}{\theta}\right]^2 \cdot \lim_{x \to 0} \frac{1}{\cos^2 3x}$$

$$= \frac{9}{5} \cdot 1^2 \cdot 1^2 = \frac{9}{5}$$

Ch2_Calc I

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2.4 Implicit Differentiation

Example $x^2 + y^2 = 1$

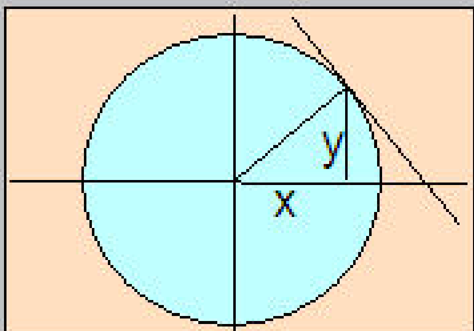
$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}1$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$



$$y = \pm\sqrt{1 - x^2}$$

$$y = \pm(1 - x^2)^{1/2}$$

$$\frac{dy}{dx} = \pm\frac{1}{2}(1 - x^2)^{-1/2}\frac{d}{dx}(1 - x^2)$$

$$\frac{dy}{dx} = \pm\frac{1}{2}\frac{1}{\sqrt{1 - x^2}}(-2x)$$

$$\frac{dy}{dx} = \frac{-x}{\pm\sqrt{1 - x^2}}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Graph

Ch2_Calc I

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2.6 Implicit Differentiation (2)

Example

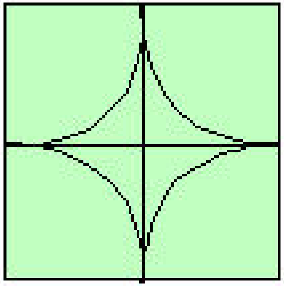
$$x^{2/3} + y^{2/3} = a^{2/3}$$

$$\frac{d}{dx}x^{2/3} + \frac{d}{dx}y^{2/3} = \frac{d}{dx}a^{2/3}$$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$x^{-1/3} + y^{-1/3}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}}$$



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2.6 Implicit Differentiation (2)

Example $x^4 + 4x^2y^2 + y^4 = 0$

$$\frac{d}{dx}x^4 + 4\frac{d}{dx}(x^2y^2) + \frac{d}{dx}y^4 = 0$$

$$4x^3 + 4\left(2xy^2 + x^2 2y\frac{dy}{dx}\right) + 4y^3\frac{dy}{dx} = 0$$

$$4x^3 + 8xy^2 + 8x^2y\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 0$$

$$(8x^2y + 4y^3)\frac{dy}{dx} = -(4x^3 + 8xy^2)$$

$$\frac{dy}{dx} = -\frac{(4x^3 + 8xy^2)}{8x^2y + 4y^3}$$

$$\frac{dy}{dx} = -\frac{x^3 + 2xy^2}{2x^2y + y^3}$$

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3.1 Related Rates

Introduction

Related Rates problems involve variables that change with respect to time.

To solve related rate problems, try the following strategy.

1) Draw a picture. Assign variables.

2) Write out the given data. Identify the unknown.

3) Write a relation between the appropriate variables

4) Differentiate with respect to time.

5) Solve for the unknown variable. Plug in the data.

6) Read the problem again. Did you answer the question?

Geometry Formulas

Examples

1) Balloon

2) Cone

3) Boat

Ch3_Calc I

File Edit Page Applications Help

Helpful Formulas

1) PYTHAGORAS' THEOREM

2) AREAS AND VOLUMES

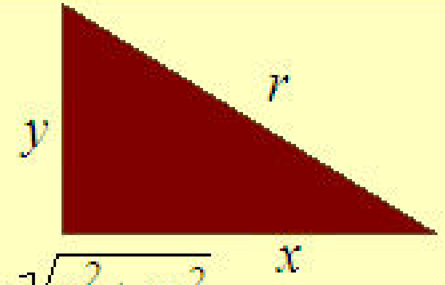
a) Box

b) Cylinder

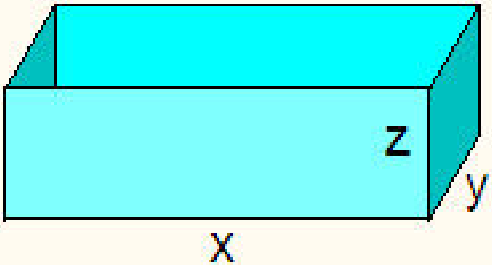
c) Sphere

d) Cone

e) Pyramid



$$r = \sqrt{x^2 + y^2}$$



Volume = xyz

Area = 2(xy+xz+yz)

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Helpful Formulas

1) PYTHAGORAS' THEOREM

2) AREAS AND VOLUMES

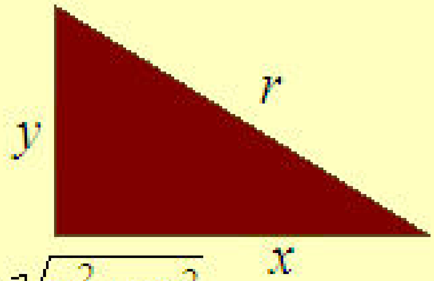
a) Box

b) Cylinder

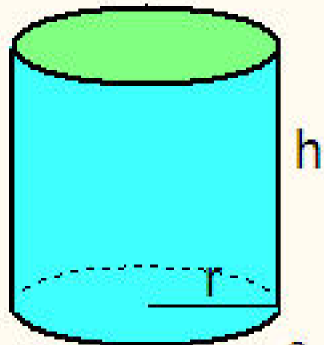
c) Sphere

d) Cone

e) Pyramid



$$r = \sqrt{x^2 + y^2}$$



$$\text{Volume} = \pi r^2 h$$

$$\text{Area} = 2 \pi r h$$

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Helpful Formulas

1) PYTHAGORAS' THEOREM

2) AREAS AND VOLUMES

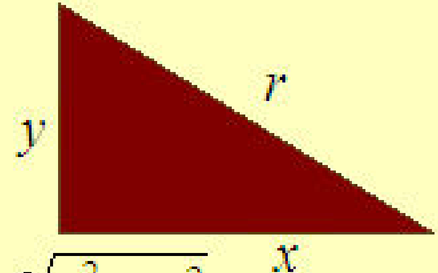
a) Box

b) Cylinder

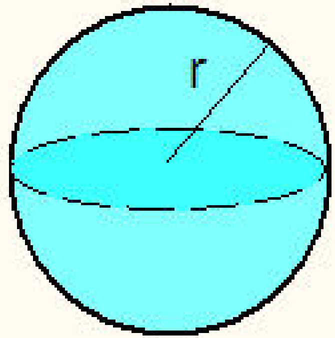
c) Sphere

d) Cone

e) Pyramid

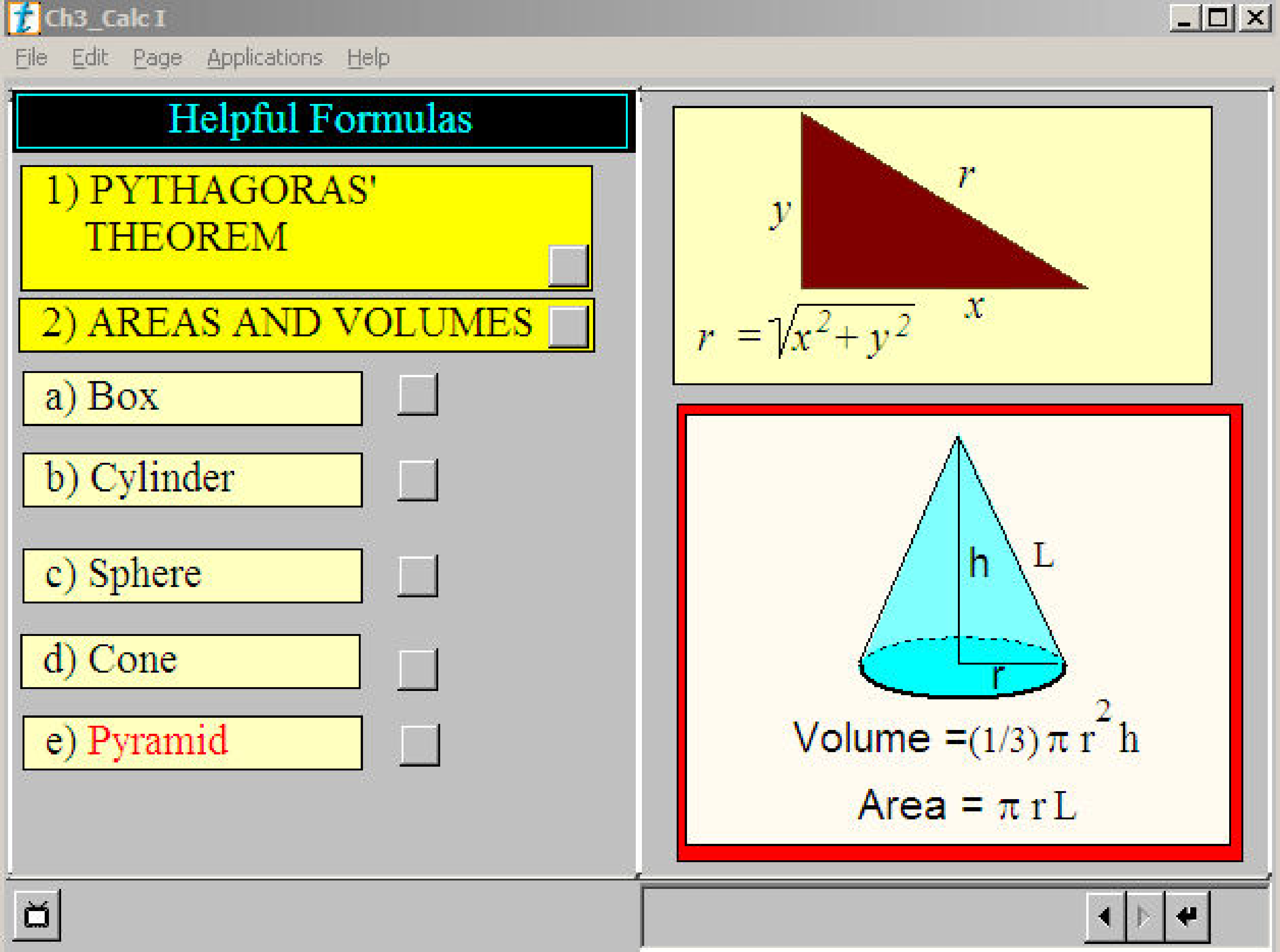


$$r = \sqrt{x^2 + y^2}$$



Volume = $(4/3)\,\pi\, r^3$

Area = $4\,\pi\, r^2$


Ch3_Calc1
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Helpful Formulas
1) PYTHAGORAS' THEOREM
2) AREAS AND VOLUMES
a) Box
b) Cylinder
c) Sphere
d) Cone
e) Pyramid
y
r
x
$r = \sqrt{x^2 + y^2}$
h
L
r
Volume = $(1/3)\,\pi\, r^2 h$
Area = $\pi\, r\, L$

Ch3_Calc I

File Edit Page Applications Help

Helpful Formulas

1) PYTHAGORAS' THEOREM

2) AREAS AND VOLUMES

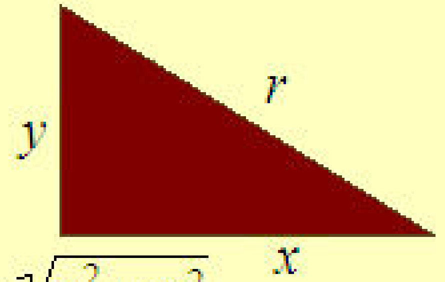
a) Box

b) Cylinder

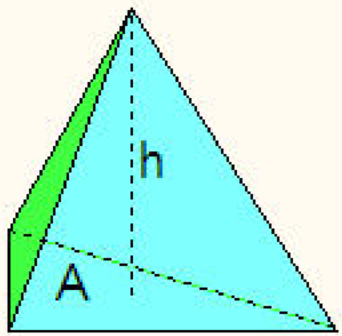
c) Sphere

d) Cone

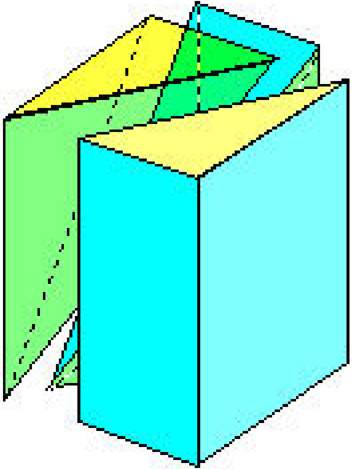
e) Tetrahedron



$$r = \sqrt{x^2 + y^2}$$



Volume = (1/3) A h



Ch3_Calc I

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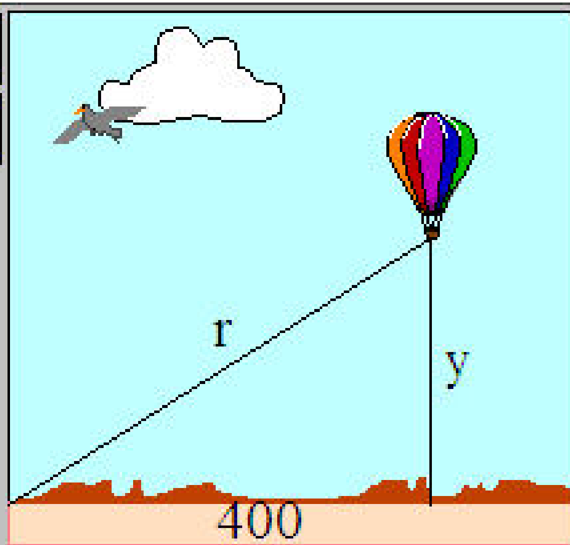
Example 1. Balloon Rising

A balloon rising straight up. The distance r from the balloon to a range finder located 400 ft from the point of lift off is increasing at a rate of 150 ft/min. How fast is the balloon rising when $r = 500$ ft?

Animate

Procedure



Solution

$$r^2 = y^2 + 400^2$$

$$\frac{d}{dt}(r^2) = \frac{d}{dt}(y^2 + 400^2)$$

$$2r\frac{dr}{dt} = 2y\frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{r}{y}\cdot\frac{dr}{dt}$$

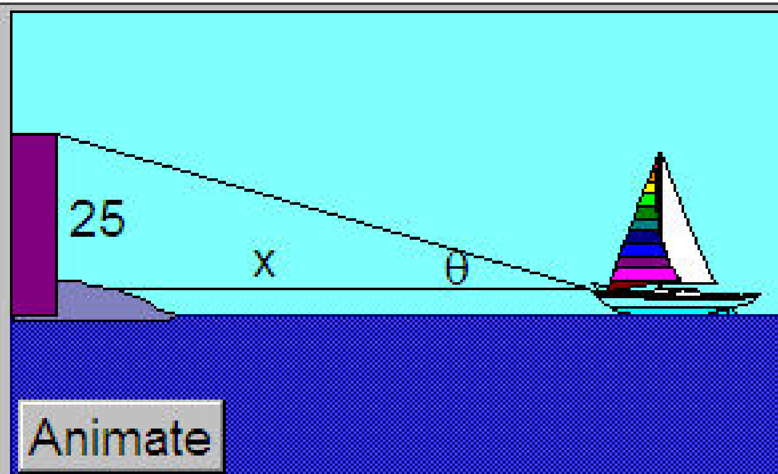
$$\left.\frac{dy}{dt}\right|_{r=500} = \frac{500}{400}\cdot 150 = 187.5$$

Ch3_Calc I

File Edit Page Applications Help

Example 2. Speed of a Boat

The angle of depression to the bow of a boat from a 25 m tower is 0.4 rad when the boat is 150 m away. If the angle is changing at 0.02 rad/s, what is the speed of the boat?



Solution

$$\frac{d\theta}{dt} = 0.02 \qquad x = 150$$

$$\frac{x}{25} = \cot\theta \qquad x = 25\cot\theta$$

$$\frac{dx}{dt} = -25\csc^2\theta\frac{d\theta}{dt}$$

When $x = 150$, $\csc\theta = \sqrt{37}$

$$\frac{dx}{dt} = -25(37)(0.02)$$

$$\boxed{\frac{dx}{dt} = 18.5}$$

Ch3_Calc I

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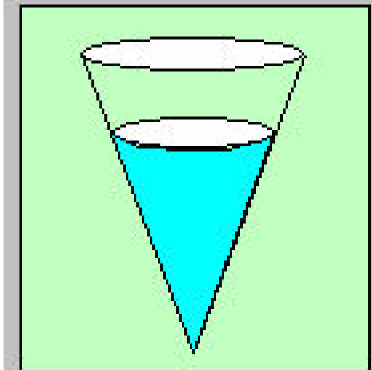
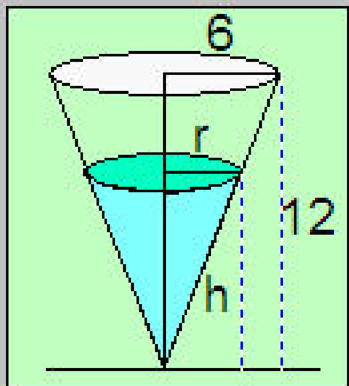
Example 2. Filling a Cone

Water is pouring into a conical cistern at the rate of 8 cu. m /min. If the height of the cistern is 12 m and the radius of the circular opening is 6 m, how fast is the water level raising when the water is 4 m deep.





Empty

Fill

Solution

Given:

$$\frac{dV}{dt} = 8$$

$$h = 4$$

$$\frac{r}{h} = \frac{6}{12}$$

$$r = \frac{h}{2}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{12}\pi h^3$$

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{4}{\pi 4^2} 8$$

$$\frac{dh}{dt} = \frac{2}{\pi} \sim 0.637$$

Ch3_Calc I

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Solution Take the first derivative and set equal to 0

$$f(x) = \frac{1}{x^2 - 4}$$

$$f'(x) = \frac{-2x}{(x^2-4)^2} = 0$$

Critical Point

$$x = 0$$

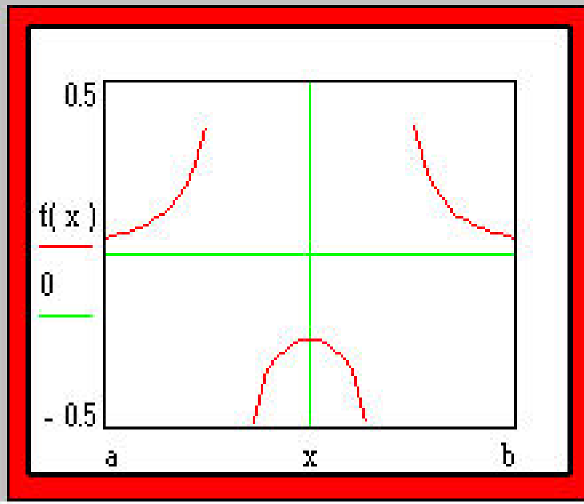
$$f''(x) = \frac{(x^2-4)^2(-2) + (2x)\,2(x^2-4)(2x)}{(x^2-4)^4}$$

$$f''(x) = \frac{2(x^2-4)[-(x^2-4) + 4x^2]}{(x^2-4)^4}$$

$$f''(x) = \frac{2(3x^2+4)}{(x^2-4)^3}$$

$$f''(0) < 0$$

Rel. Max. at $x = 0$



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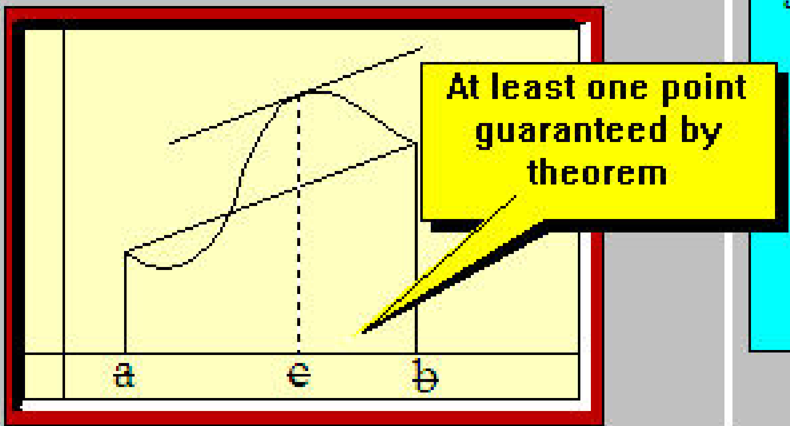
Mean Value Theorem

Theorem:

If f(x) is continuous on [a,b], and differentiable on (a,b), then, there exists at least one point c in (a,b) where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Example

$f(x) = x^3 - 3x$ on $[0,3]$.

$a = 0 \qquad f(a) = 0$

$b = 3 \qquad f(b) = 18$

$m = \dfrac{18 - 0}{3 - 0} \qquad m = 6$

$f'(x) = 3x^2 - 3$

$f'(c) = 3c^2 - 3 = 6$

$3c^2 = 9$

$c^2 = 3$

$c = \sqrt{3}$

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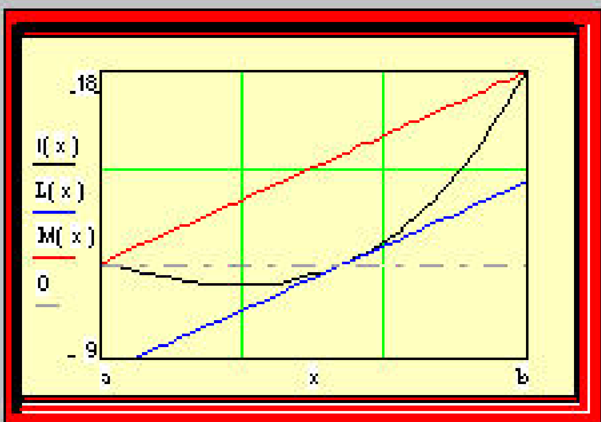
Mean Value Theorem

Theorem:

If f(x) is continuous on [a,b], and differentiable on (a,b), then, there exists at least one point c in (a,b) where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Example

$f(x) = x^3 - 3x$ on $[0,3]$.

$a=0$ $f(a)=0$

$b=3$ $f(b)=18$

$$m = \frac{18 - 0}{3 - 0} \qquad m=6$$

$f'(x) = 3x^2 - 3$

$f'(c) = 3c^2 - 3 = 6$

$3c^2 = 9$

$c^2 = 3$

$c = \sqrt{3}$

Graph Maple

Field "f3" of Page "MVT" 5430,825 16 of 22

Second Derivative Test

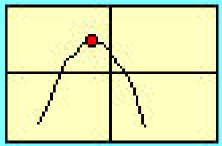
$y = f(x)$

Problem. Find the relative extrema

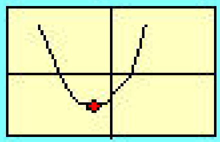
Step 1. Set $f'(x)=0$.
Solve to find the critical points c.

Step 2. Evaluate f'' at $x=c$.

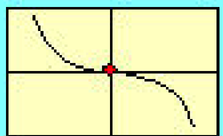
$f''(c) < 0$. Rel. Maximum

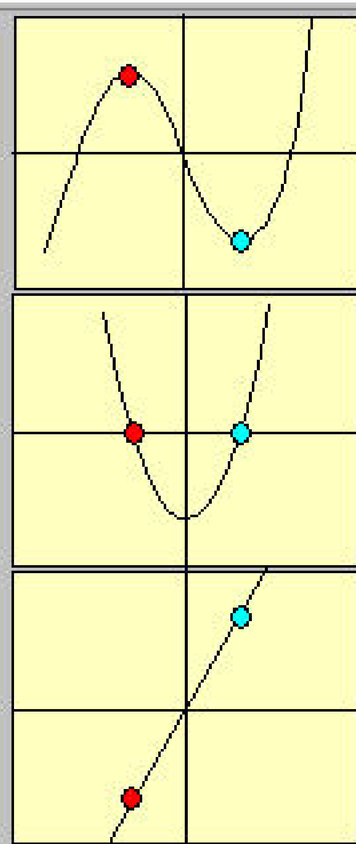


$f''(c) > 0$. Rel. Minimum



$f''(c) = 0$. Inflection Point ??





Picture Diff2 Ex.

Ch3_Calc I

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Solution Take the first derivative and set equal to 0

$$f(x) = \frac{1}{x^2 + 4}$$

$$f'(x) = \frac{-2x}{(x^2+4)^2} = 0$$

Critical Point

$x = 0$

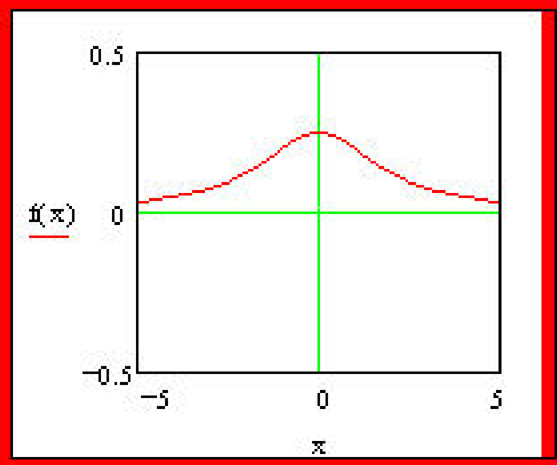
$$f''(x) = \frac{(x^2+4)^2(-2) + (2x)\,2(x^2+4)(2x)}{(x^2+4)^4}$$

$$f''(x) = \frac{2(x^2+4)[-(x^2+4) + 4x^2]}{(x^2+4)^4}$$

$$f''(x) = \frac{2(3x^2-4)}{(x^2+4)^3}$$

$f''(0) < 0$

Rel. Max. at $x = 0$



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3.4 Optimization

INTRODUCTION

Goal: To find the maximum or the minimum of some function in certain domain.

Strategy

1) Draw a picture. Assign variables.

2) Identify the Objective Function

3) Identify the Constraint

4) Use the Constraint to eliminate one variable

5) Apply the derivative tests

6) Reread the problem. Answer the question

Examples:

1) Folding a box

2) Cutting a beam

3) Can of soup

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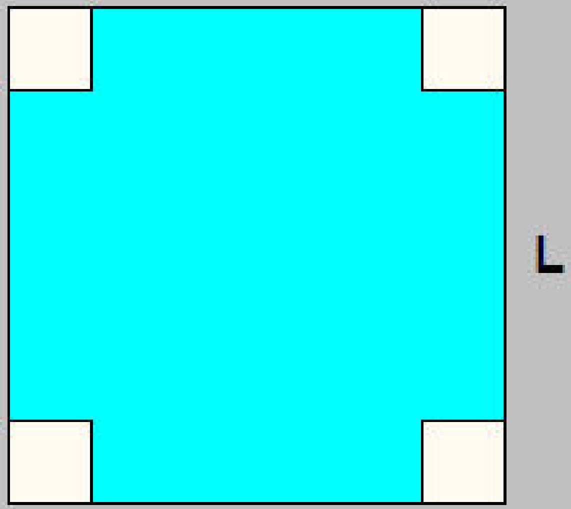
Example 1. Folding a Box

An open box is to be made by cutting out the corners of a square sheet **a** cm in diameter and folding up the sides. What is the maximum volume?

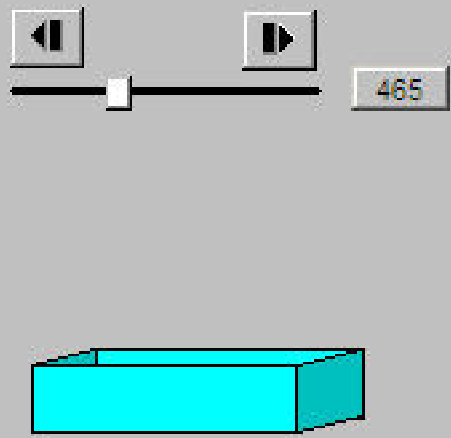


a = 30.00 cm

L = 20.00 cm

H = 5.00 cm

Volume 2000.00 cm^3



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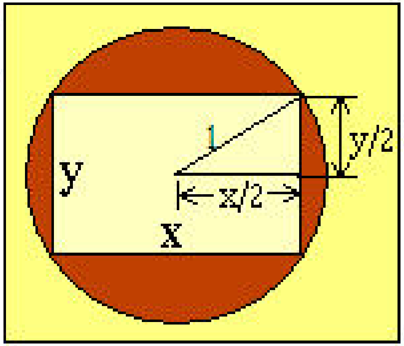
Example 2. Cutting Out a Beam

A beam of rectangular cross-section is to be cut from a log of radius 1. Find the dimensions of the cross-section that will maximize the area.

Solution



$$(y/2)^2 = 1-(x/2)^2$$

$$y = 2\sqrt{1-(x/2)^2}$$

$$y = (4 - x^2)^{1/2}$$

$$A = xy = x\ (4 - x^2)^{1/2}$$

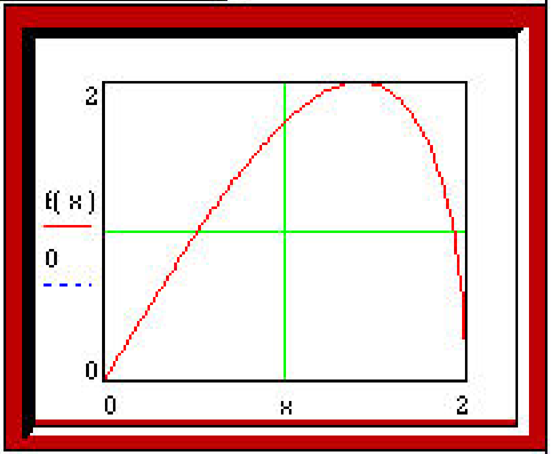
Graph

$$DA := -2\cdot\frac{\left(-2 + x^2\right)}{\sqrt{\left(4 - x^2\right)}}$$

$$x = \sqrt{2}$$

$$A(\sqrt{2}) = 2$$
$$A(0) = 0$$
$$A(2) = 0$$

Ans: $x = \sqrt{2}\ ,\ y = \sqrt{2}$



Ch3_Calc I

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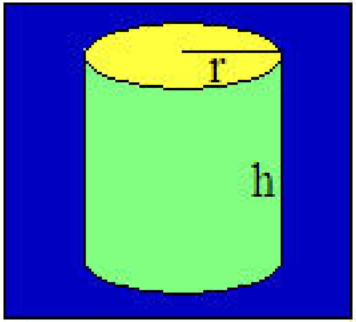
Example 3. Can of Soup

Find the dimensions of a can of soup of minimum surface area, if the volume is 16π.



Solution

$$V = \pi r^2 h = 16$$

$$A = 2\pi r^2 + 2\pi r h$$

$$h = 16/(\pi r^2)$$

$$A = 2\pi r^2 + 2\pi r(16/\pi r^2)$$

$$A = 2\pi r^2 + 32/r$$

graph2

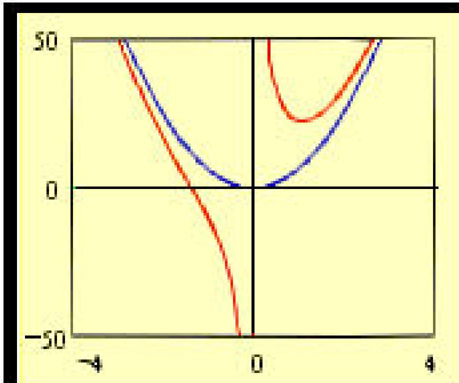
$$A' = 4\pi r - 32/r^2 = 0$$

$$4\pi r = 32/r^2 \qquad r^3 = 32/4\pi = 8/\pi$$

$$r = 2/\pi^{1/3} \qquad h = 16\pi^{2/3}/(4\pi)$$

$$h = 4/\pi^{1/3} \qquad h = 2r$$

if $r > 0$ $\qquad A'' = 4\pi + 64/r^3 > 0$



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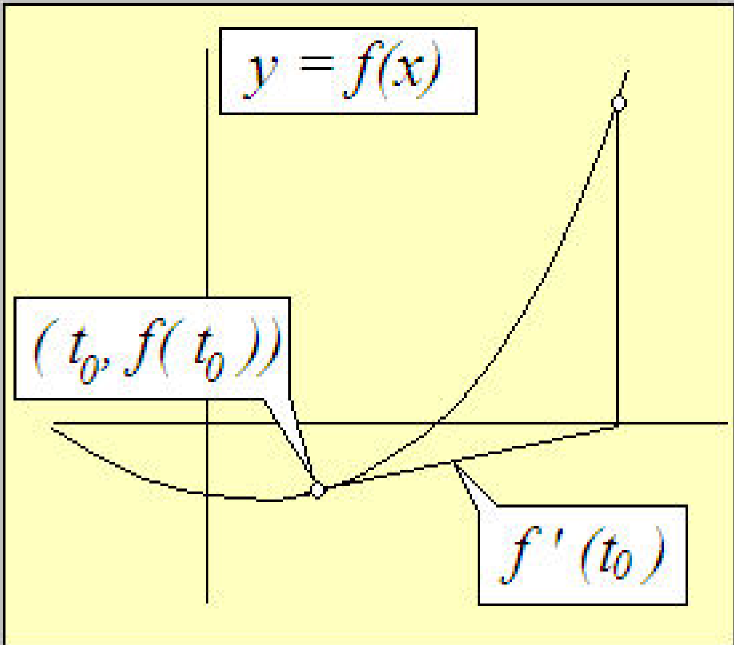
2.6 Newton's Method



$$\frac{y - f(t_0)}{x - t_0} = f'(t_0)$$

$$y - f(t_0) = f'(t_0)(x - t_0)$$

$$\frac{-f(t_0)}{f'(t_0)} = t_1 - t_0$$

$$t_1 = t_0 - \frac{f(t_0)}{f'(t_0)}$$

$$t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}$$

Ch4_Calc I

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4.2 Differential Equations

Introduction: Equations of the form

$$dy/dx = g(x,y)$$

are called first order differential

Case $dy/dx = g(x)$

Solution: $y = \int g(x)\,dx$

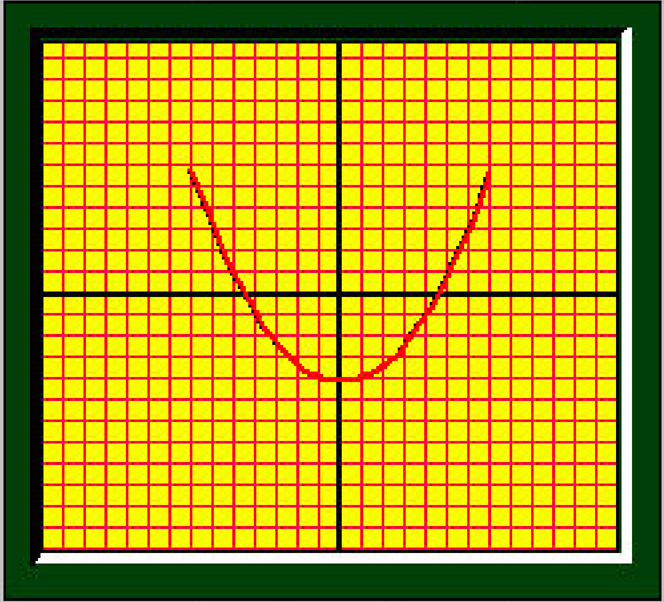
Ex. $dy/dx = 2x$

Solution: $y = \int 2x\,dx = x^2 + c$

We get a family of solutions labeled by c. To determine c we need initial conditions such as:

$$y(0) = -4$$

$y = x^2 - 4$



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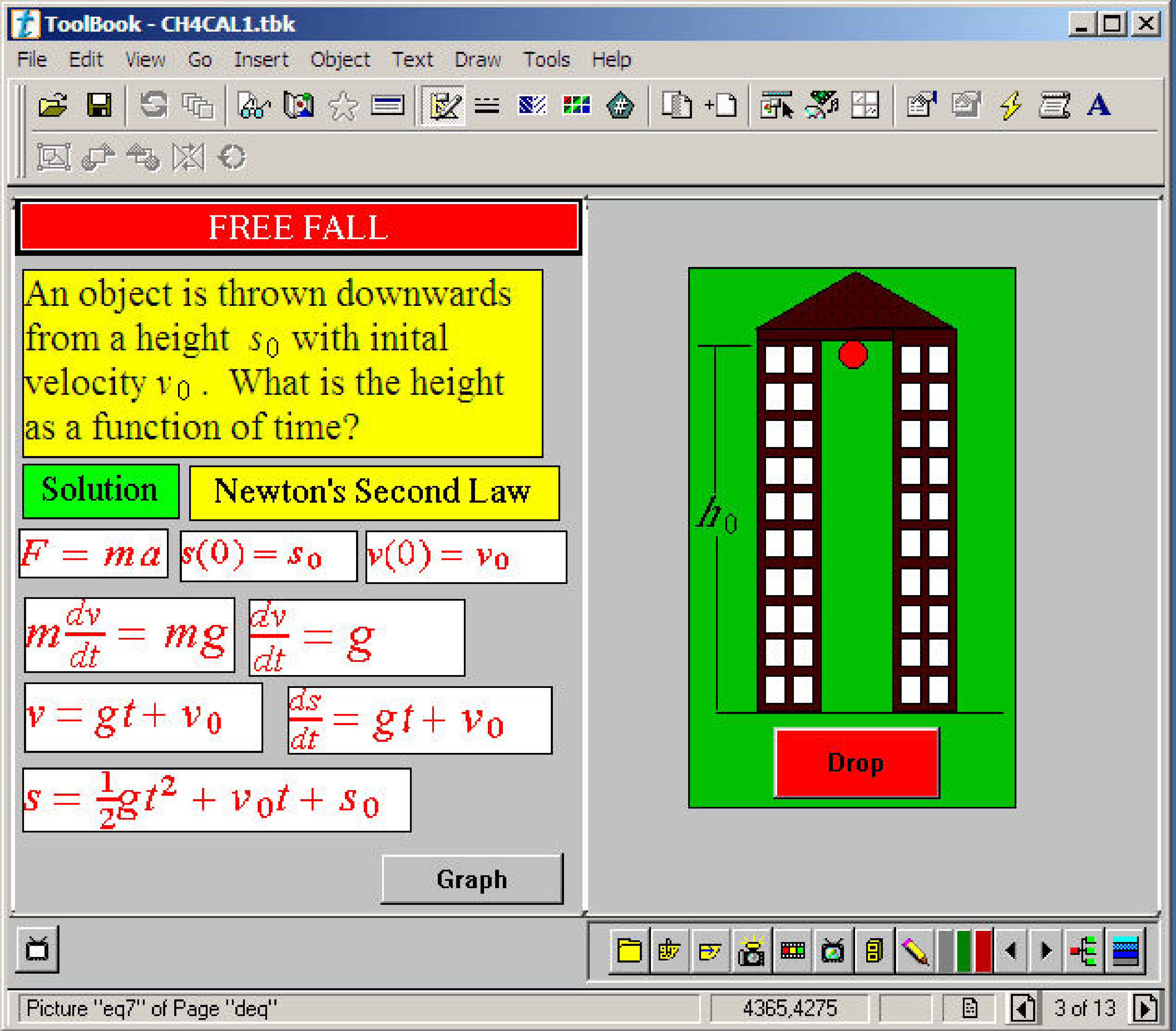
ToolBook - CH4CAL1.tbk
File Edit View Go Insert Object Text Draw Tools Help
FREE FALL
An object is thrown downwards from a height s_0 with inital velocity v_0. What is the height as a function of time?
Solution
Newton's Second Law
$F = ma$
$s(0) = s_0$
$v(0) = v_0$
$m\frac{dv}{dt} = mg$
$\frac{dv}{dt} = g$
$v = gt + v_0$
$\frac{ds}{dt} = gt + v_0$
$s = \frac{1}{2}gt^2 + v_0 t + s_0$
Graph
h_0
Drop
Picture "eq7" of Page "deq"
4365,4275
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Riemann Sums

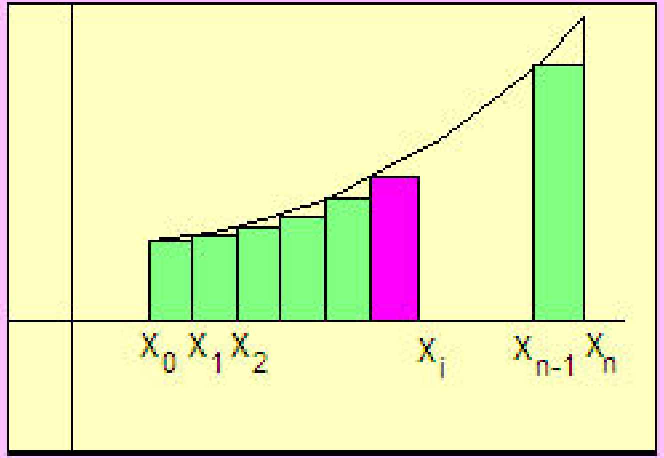
Area Under a Curve

How do we find the area under the function f(x) between x = a and x = b?

Partition the interval [a,b] into n equal pieces by choosing points
$a = x_0 < x_1 < \cdots < x_i < \cdots < x_n = b$

Draw vertical lines at the partition points. Approximate the area by rectangles.

$$\Delta x = \frac{b-a}{n} \qquad x_i = a + i\Delta x$$



$$\Delta A = f(x_i)\ \Delta x$$

$$A = \int_a^b f(x)\,dx \equiv \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Fundamental Theorem of Calculus

Theorem (FTC)

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

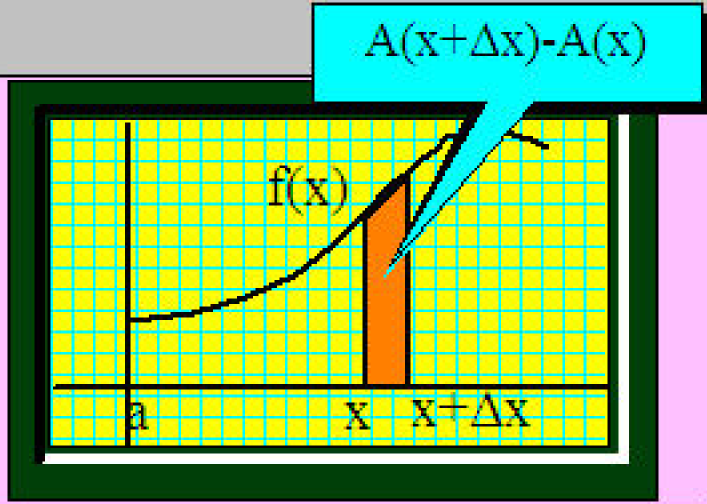
where,

$$F'(x) = f(x)$$

Discussion: Let A(x) denote the area under f(x) in the interval [a,x].

$$A(x) = \int_a^x f(t)\,dt$$

$$\Delta A = A(x+\Delta x) - A(x) \approx f(x)\,\Delta x$$



$$\frac{dA}{dx} = \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = f(x)$$

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Ex. $\dfrac{d}{dx}\displaystyle\int_2^x \sin t^2\,dt = \sin x^2$

Ch4_Calc I

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Fundamental Theorem of Calculus

Theorem (FTC)

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

$$\int_a^b f(x)dx = F(b) - F(a)$$

where,

$$F'(x) = f(x)$$

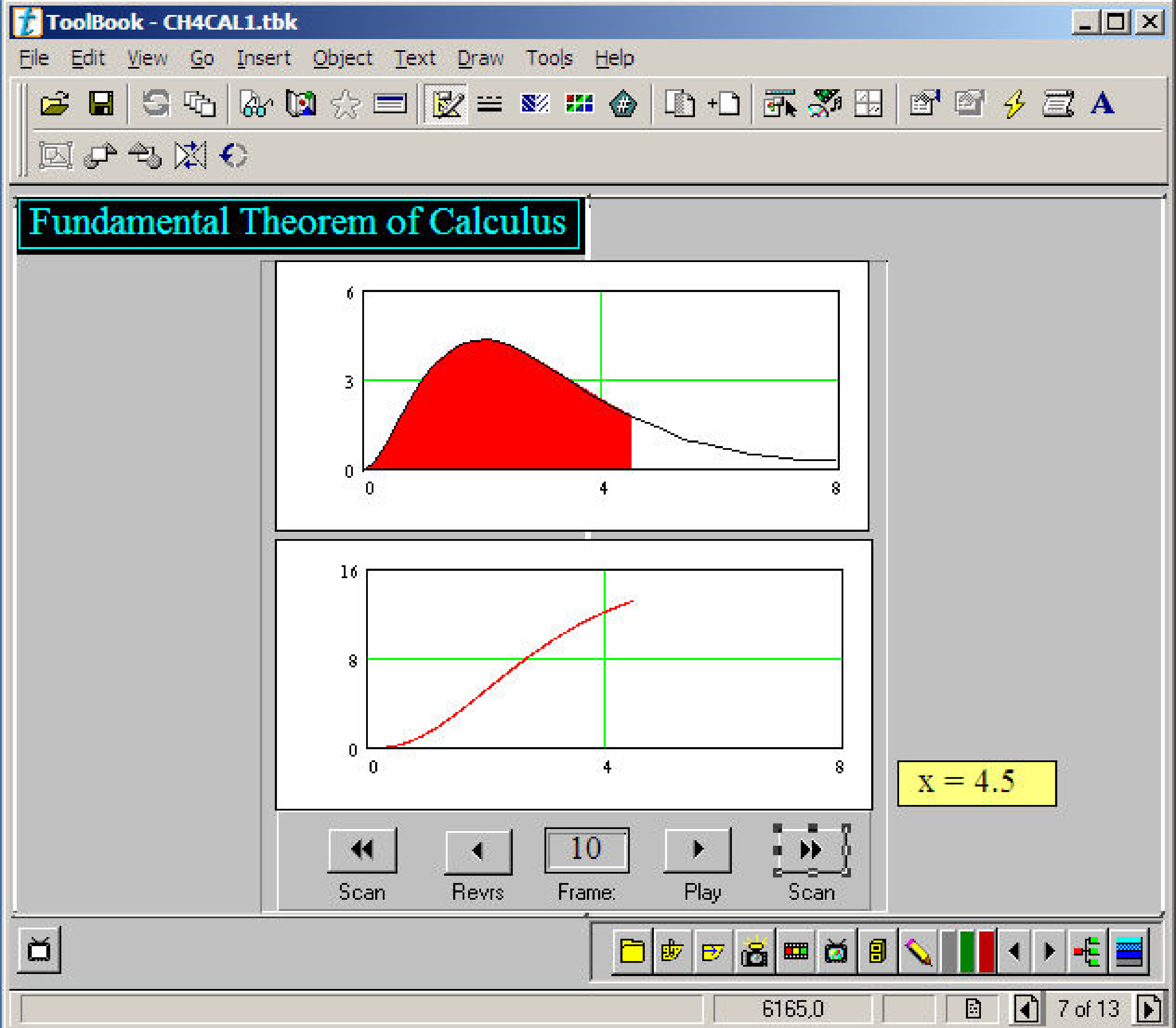
(cont) $$\int_a^x f(t)dt = F(x) + C$$

$$\int_a^b f(x)dx = F(b) + C$$

$$\int_a^a f(x)dx = F(a) + C = 0$$

$$C = -F(a)$$

$$\int_a^b f(x)dx = F(b) - F(a)$$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Fundamental Theorem of Calculus
6
3
0
0
4
8
16
8
0
0
4
8
x = 4.5
10
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Ch4_Calc I

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Numerical Methods

Trapezoidal Rule

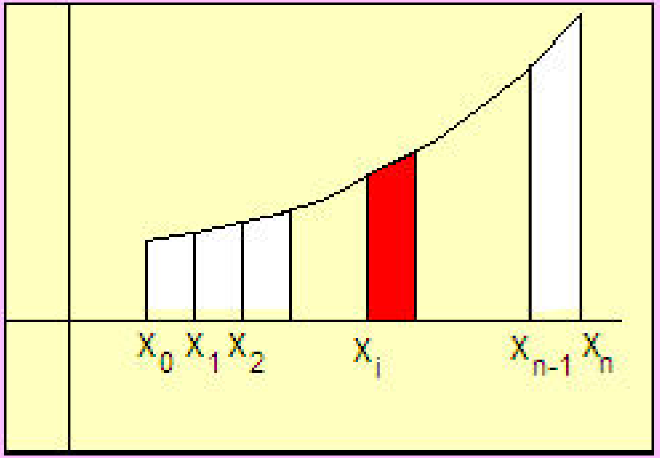
Let A be the area under the function f(x) between x = a and

Partition the interval [a,b] into n equal pieces by choosing points $a = x_0 < x_1$ <---- $< x_i$ <---- $< x_n = b$

Draw vertical lines at the partition points. Approximate the area by trapezoids.

$$\Delta x = \frac{b - a}{n} \qquad x_i = a + i\,\Delta x$$

$$A = [f(x_0) + \dots 2f(x_i) + \dots f(x_n)]\frac{\Delta x}{2}$$



$$A = \sum 2f_i(x\ )\frac{\Delta x}{2} + [f(x_0\ ) + f(x_0\ )]\frac{\Delta x}{2}$$

$$A = \sum f(x_i)\,\Delta x + [f(a) + f(b)]\frac{\Delta x}{2}$$

Maple

Ch5_Calc I

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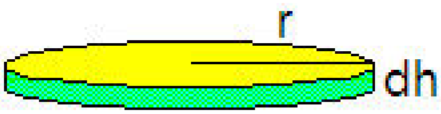
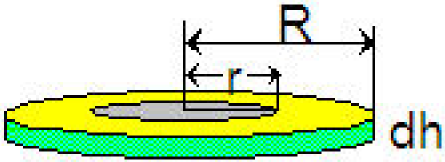
Volumes of Revolution

Disk



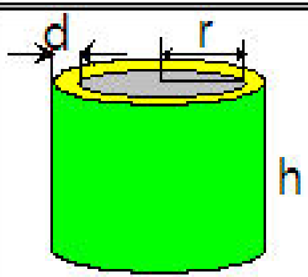
$dV = \pi r^2 dh$

Washer



$dV = \pi (R^2 - r^2)$

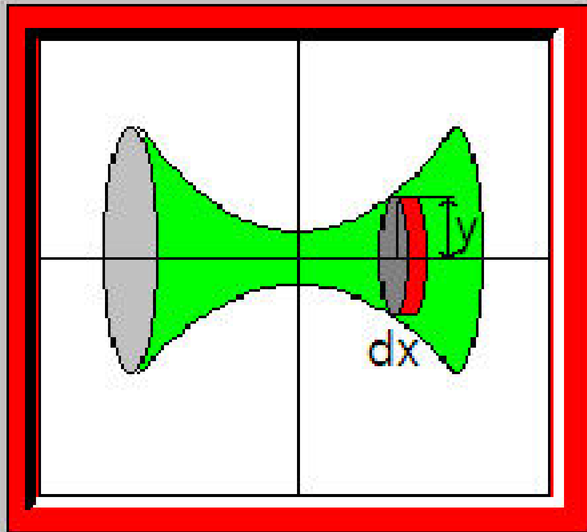
Shell



$dV = 2\pi rhdr$

Example 1

Find the volume generated by rotating the area bounded by $f(x) = x^2+1$, $x = -1$, $x = 1$, and $y = 0$, about the x-axis.



$dV = \pi y^2 dx$

Ch5_Calc I

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Volumes of Revolution

$$V = 2\pi \int_0^1 y^2 \, dx$$

$$V = 2\pi \int_0^1 (x^2 + 1)^2 \, dx$$

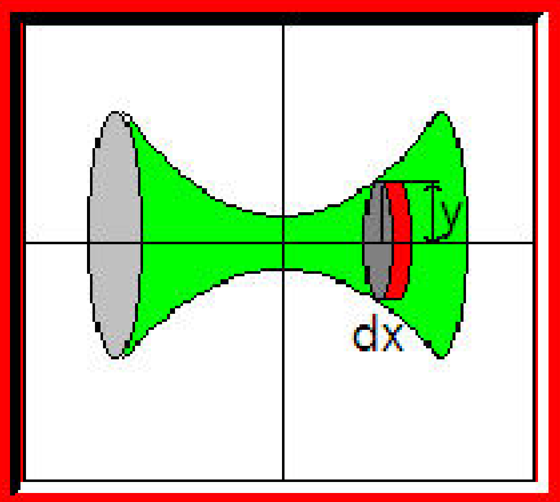
$$V = 2\pi \int_0^1 (x^4 + 2x^2 + 1) \, dx$$

$$V = 2\pi \left[(1/5)x^5 + (2/3)x^3 + x\right]\Big|_0^1$$

$$V = 2\pi \,[1/5 + 2/3 + 1] = 28$$

Example 1

Find the volume generated by rotating the area bounded by $f(x) = x^2+1$, $x = -1$, $x = 1$, and $y = 0$, about the x-axis.



$$dV = \pi \, y^2 \, dx$$

Pac_3D

Continue

Ch5_Calc I

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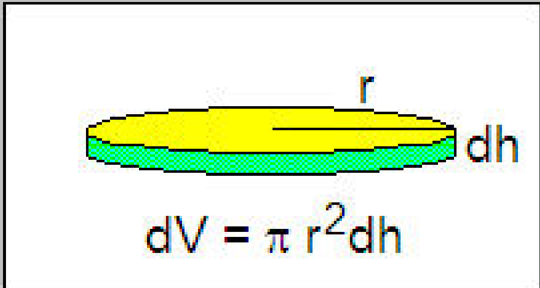
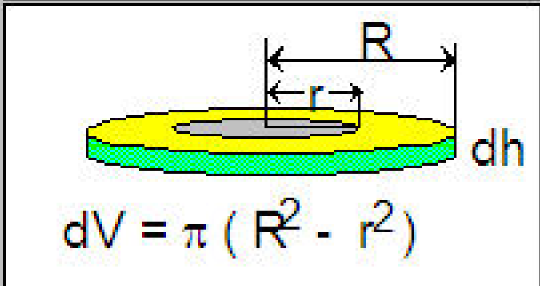
Volumes of Revolution

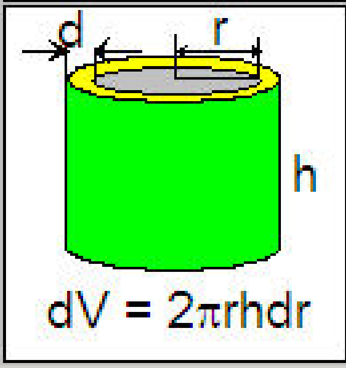
Disk



Washer

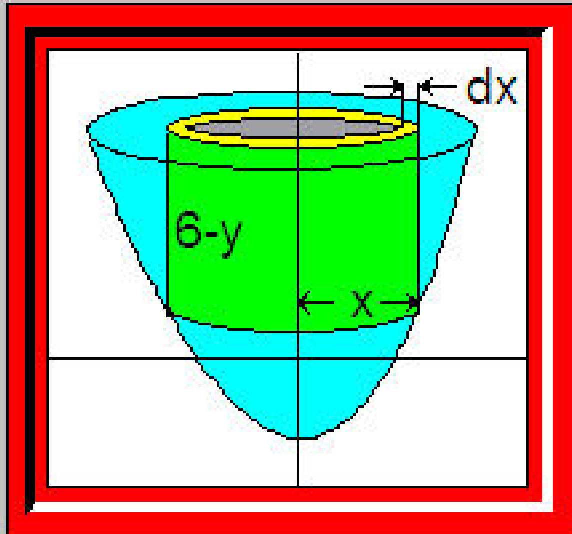


Shell



Example 2.

Find the volume generated by rotating the graph bounded by $f(x) = x^2 - 3$, $x = 0$, $y = 6$, around the y-axis.



$dV = 2\pi x(6-y)dx$.

Pac_3D

Ch5_Calc I

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Volumes of Revolution

$$V = 2\pi \int_0^3 x(6 - y)dx$$

$$V = 2\pi \int_0^3 x(6 - x^2 + 3)dx$$

$$V = 2\pi \int_0^3 (9x - x^3)$$

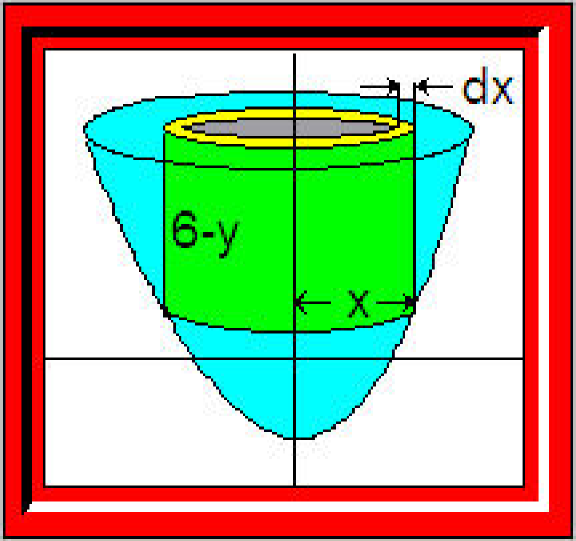
$$V = 2\pi\left(\frac{9x^2}{2} - \frac{x^4}{4}\right)\Bigg|_{x=0}^{x=3}$$

$$V = 2\pi\left[\frac{9(3)^2}{2} - \frac{3^4}{4}\right]$$

$$V = 2\pi 81\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{81\pi}{2}$$

Example 2.

Find the volume generated by rotating the graph bounded by f(x) = x^2 - 3, x = 0, y = 6, around the y-axis.



dV = 2πx(6-y)dx.

Pac_3D

Ch5_Calc I

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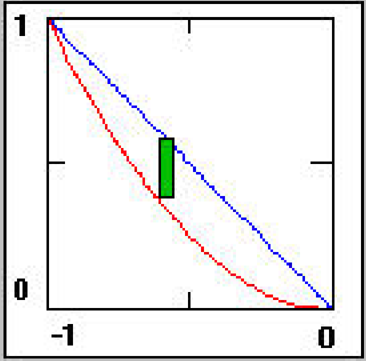
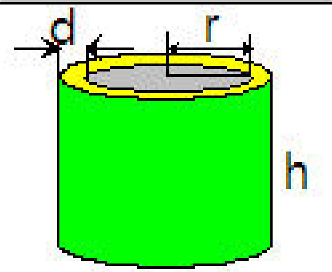
Volumes of Revolution (2)

Example 3

Find the volume generated by rotating the area bounded by $f(x) = -x^2$ and $g(x) = x$ about the y-axis.





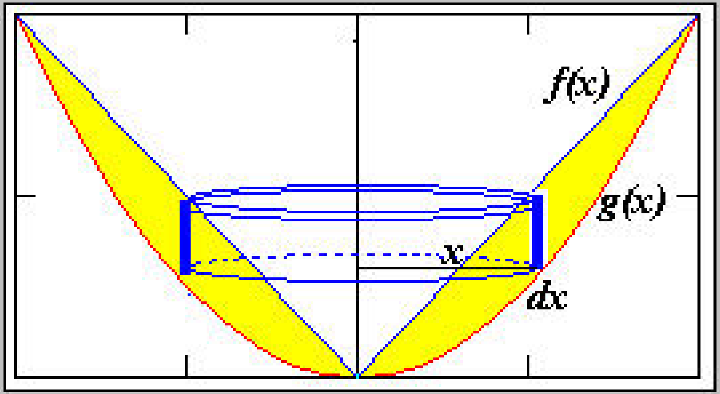
$dV = 2\pi r h dr$

Set $x^2 = -x$

$x^2 + x = 0$

$x(x+1) = 0$

$x = 0, x = -1$



$dV = 2\pi x\,[\,f(x) - g(x)\,]\,dx$

Sum the volume of the shells as x ranges from -1 to 0.

$$V = 2\pi \int_{-1}^{0} x\,[f(x) - g(x)]\,dx$$

Maple

Sphere Tank Bell

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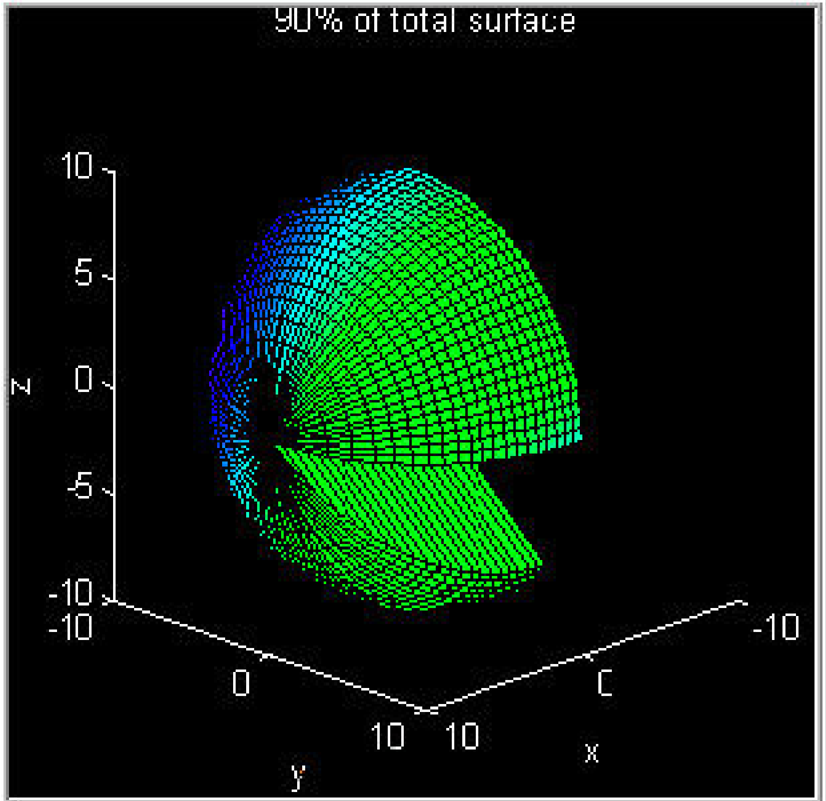
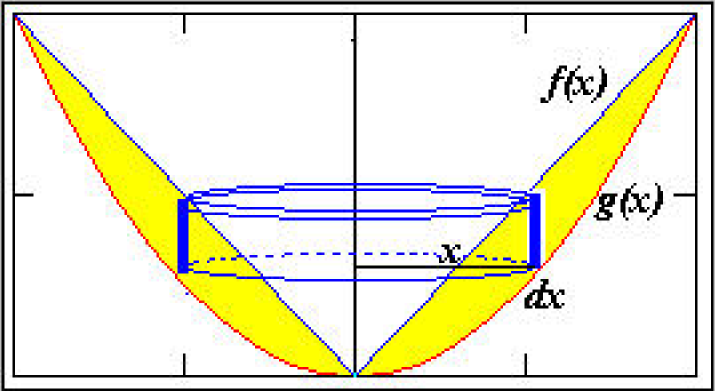
Volumes of Revolution (2)





$dV = 2\pi x\,[\,f(x) - g(x)\,]\;dx$

Sum the volume of the shells as x ranges from -1 to 0.

$$V = 2\pi \int_{-1}^{0} x[f(x) - g(x)]dx$$

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Sphere Tank Bell

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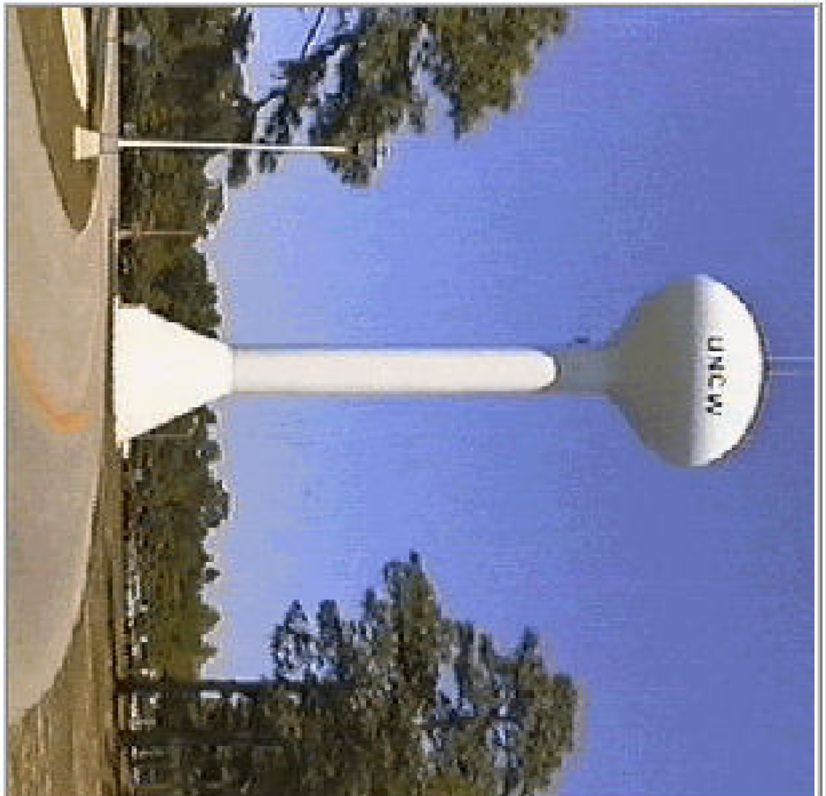
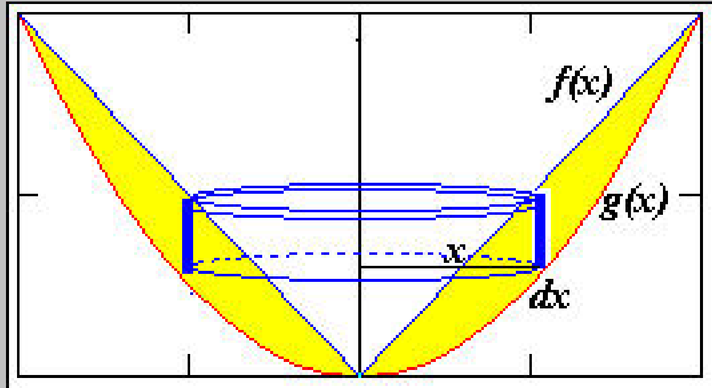
Volumes of Revolution (2)





$dV = 2\pi x\,[\,f(x) - g(x)\,]\;dx$

Sum the volume of the shells as x ranges from -1 to 0.

$$V = 2\pi \int_{-1}^{0} x[f(x) - g(x)]dx$$

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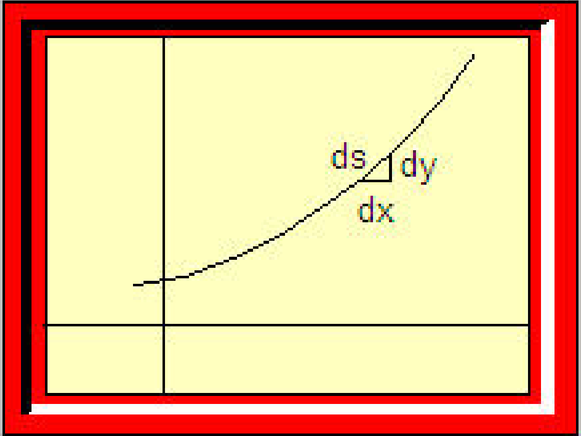
Arc Length



$$ds^2 = dx^2 + dy^2$$

$$ds^2 = \left[1 + \left(\frac{dy}{dx}\right)^2 \right] dx^2$$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

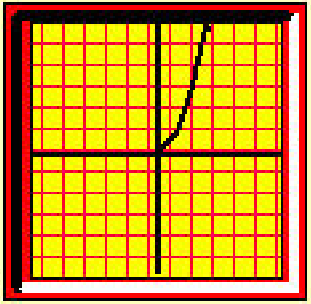
Example: Find the arc length of

$y = x^{3/2} \qquad 0 \le x \le 4$

$$ds^2 = dx^2 + dy^2 \qquad dy = \frac{3}{2}x^{1/2}dx$$

$$ds^2 = dx^2 + \left[\frac{3}{2}x^{1/2}dx\right]^2$$

$$ds^2 = dx^2 + \frac{9}{4}x\,dx^2$$



$$ds = \sqrt{1 + \frac{9}{4}x}\, dx$$

$$s = \int_0^4 \sqrt{1 + \frac{9}{4}x}\, dx$$

$$s = \frac{2}{3} \cdot \frac{4}{9}\left[1 + \frac{9}{4}x\right]^{3/2}$$

$$s = \frac{8}{27}\left[10^{3/2} - 1\right]$$

Ch5_Calc I

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Center of Mass

$$M \quad m_1 + m_2$$

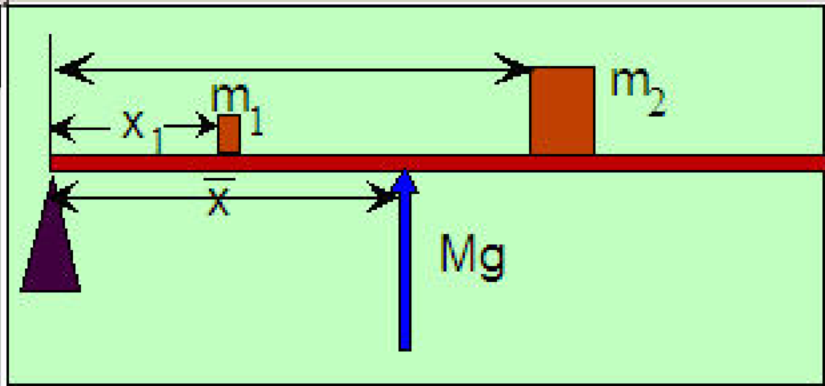
$$M g \bar{x} = m_1 g x_1 + m_2 g x_2$$

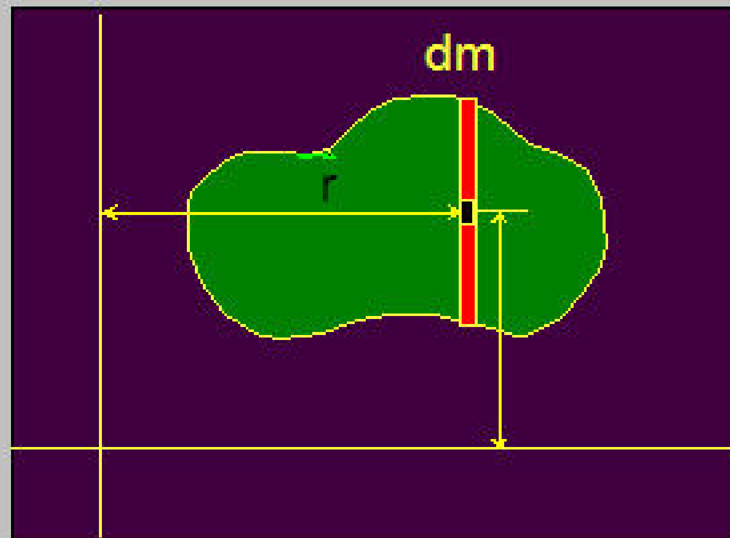
$$M \bar{x} = m_1 x_1 + m_2 x_2$$

$$\bar{x} \quad \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

$$\bar{r} = \frac{\int_A r\,dm}{\int_A dm}$$





r = Arm lever of c.m. of strip