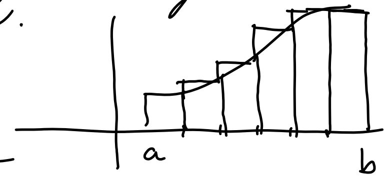
Areas and Distances

5.1 Areas and Distances

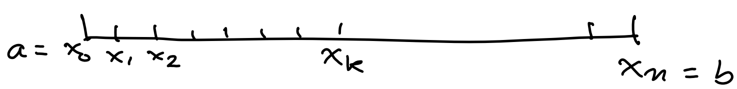
Let $f(x)$ be a continuous function on a closed interval $[a,b]$. For now assume that $f(x) > 0$ on this interval. We are interested in finding the area under the graph.



Following the ideas of the "method of exhaustion" which goes all the way back to Archimedes, we proceed as follows

1. We create a partition of the interval $[a,b]$ by dividing the segment into n pieces. We will assume that all these pieces are of equal size Δx.

$$\boxed{\Delta x = \frac{b-a}{n}}$$



The set of points of the partition will be labeled

$$a = x_0 < x_1 < x_2 < \cdots < x_k < \cdots < x_{n-1} < x_n = b$$

2. We approximate the area under the by the sum of the areas of rectangles with bases of size Δx and heights given the value of the function

$$A_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_k)\Delta x + \cdots + f(x_n)\Delta x$$

$$= \sum_{k=1}^{n} f(x_k)\Delta x \qquad \boxed{x_k = a + k\Delta x}$$

3. Take the limit as $n \to \infty$

$$\boxed{A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x}$$

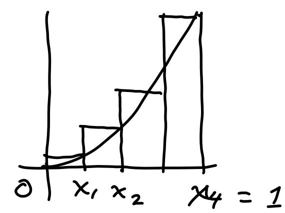
Example: Find the area under the graph $f(x) = x^2$ for $x \in [0, 1]$

Solution:

a) Approximation with $n = 4$

$a = 0 \quad \Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$

$b = 1$



$x_k = 0 + k\,\Delta x$

$= \frac{k}{4}$

$x_0 = 0, \quad x_1 = \frac{1}{4}, \quad x_2 = \frac{2}{4}, \quad x_3 = \frac{3}{4}, \quad x_4 = \frac{4}{4}$

$$A_4 = \sum_{k=1}^{4} f(x_k)\,\Delta x = [f(x_1) + f(x_2) + f(x_3) + f(x_4)]\,\Delta x$$

$$A_4 = \left[\left(\frac{1}{4}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + \left(\frac{4}{4}\right)^2\right]\frac{1}{4}$$

$$= \left[\frac{1}{16} + \frac{4}{16} + \frac{9}{16} + \frac{16}{16}\right] \cdot \frac{1}{4} = \frac{30}{(16)(4)} = \frac{15}{32}$$

b) Exact area.

$\Delta x = \frac{1}{n} \qquad x_k = 0 + k\,\Delta x = \frac{k}{n}$

$$A = \lim_{n \to \infty} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\Delta x = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{k^2}{n^2}\,\frac{1}{n}$$

$$= \lim_{n \to \infty} \frac{1}{n^3} \sum_{k=1}^{n} k^2$$

To evaluate this limit we need the following formula that we state without proof

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\therefore A = \lim_{n \to \infty} \frac{n(n+1)(2n+1)}{6n^3} = \lim_{n \to \infty} \frac{2n^3 + \cdots}{6n^3}$$

$$A = \frac{2}{6} = \frac{1}{3}$$

Example: Generalize the previous problem and find the area under $f(x) = x^2$ in the interval $[0, b]$

Sol:

$\Delta x = \frac{b-a}{n} = \frac{b}{n}$ $\qquad x_k = a + k\Delta x = b\frac{k}{n}$

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{b^2 k^2}{n^2} \cdot \frac{b}{n}$$

$$= \lim_{n\to\infty} \frac{b^3}{n^3} \sum_{k=1}^{n} k^2 = \lim_{n\to\infty} b^3 \frac{n(n+1)(2n+1)}{6n^3}$$

$A = \frac{b^3}{3}$. This formula was known by Archimedes!

Distance

If $v(t)$ represents the velocity of a particle at time t, then the area under the curve represents the distance traveled.

5.1c Areas and Distances. (Cont.)

Summary:

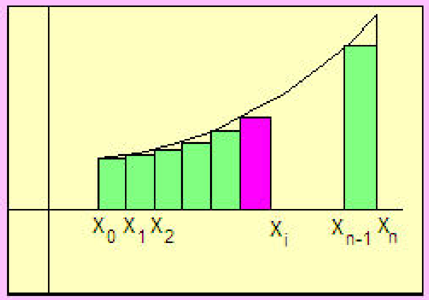
Area Under a Curve

How do we find the area under the function f(x) between x = a and x = b?

Partition the interval [a,b] into n equal pieces by choosing points $a = x_0 < x_1 < \cdots < x_i < \cdots < x_n = b$

Draw vertical lines at the partition points. Approximate the area by rectangles.

$$\Delta x = \frac{b-a}{n} \qquad x_i = a + i\Delta x$$



$$\Delta A = f(x_i)\,\Delta x$$

$$A = \int_a^b f(x)\,dx \equiv \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Numerical Methods

Trapezoidal Rule

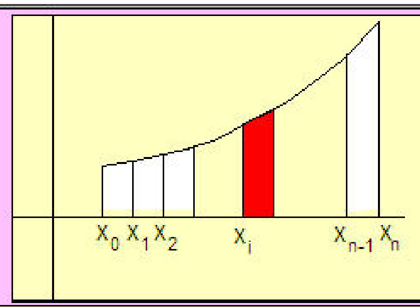
Let A be the area under the function f(x) between x = a and

Partition the interval [a,b] into n equal pieces by choosing points $a = x_0 < x_1 < \cdots < x_i < \cdots < x_n = b$

Draw vertical lines at the partition points. Approximate the area by trapezoids.

$$\Delta x = \frac{b-a}{n} \qquad x_i = a + i\,\Delta x$$

$$A = [f(x_0) + \dots 2f(x_i) + \dots f(x_n)]\frac{\Delta x}{2}$$



$$A = \Sigma\, 2f_i(x)\frac{\Delta x}{2} + [f(x_0) + f(x_0)]\frac{\Delta x}{2}$$

$$A = \Sigma\, f(x_i)\,\Delta x + [f(a) + f(b)]\,\frac{\Delta x}{2}$$

Maple

5.2 Definite Integrals

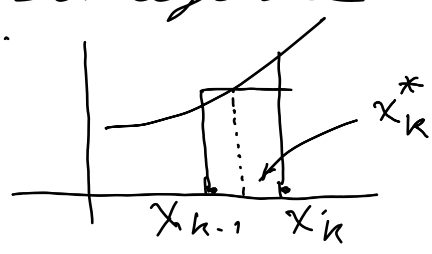
Following the ideas introduced in the previous section, we give the following definition

__Definition__: Let $f(x)$ be a continuous function on $[a,b]$.
Let $a = x_0 < x_1 < x_2 < \cdots < x_k < \cdots < x_{n-1} < x_n = b$ be a partition of $[a,b]$
Denote by Δx_k the length of the subinterval $[x_{k-1}, x_k]$ so that
$\Delta x_k = x_k - x_{k-1}$.
For each subinterval $[x_{k-1}, x_k]$ pick a point $x_k^* \in [x_{k-1}, x_k]$.
Then, the __indefinite integral__ $\int_a^b f(x)\,dx$ is defined by the formula

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k.}$$

The sum on the right hand side
$\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$ is called a __Riemann sum__.

It was shown by Riemann in the 1800's that under the assumption of $f(x)$ being a continuous function on $[a,b]$ the limit of the sum exists regardless of the choice of the partitions and of the choice of x_k^*.
In such case we say that the definite integral converges and the function $f(x)$ is integrable on the interval $[a,b]$.



Note.

1. If the partition is equally spaced as in section 1, then
$$\Delta x_k = \Delta x = \frac{b-a}{n}$$

2. If $f(x) > 0$ on $[a,b]$ then $\int_a^b f(x)\,dx$ gives the area under the graph.
If $f(x) < 0$ on $[a,b]$ then $\int_a^b f(x)\,dx$ gives the negative of area.

3. When computing the Riemann sums, it is convenient to choose x_k^* to be left (or right) endpoints of the subintervals or the midpoints (Midpoint rule).

4. The following formulas are useful in computing Riemann sums

a) $\displaystyle\sum_{k=1}^{n} 1 = n$

b) $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

c) $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

d) $\displaystyle\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$

Formulas involving higher order powers or other functions like $\sin(k)$, e^k can be obtained using Maple.

To remove some of the mystery we use recursion to derive the formula for $\sum k^2$ in the appendix to this section.

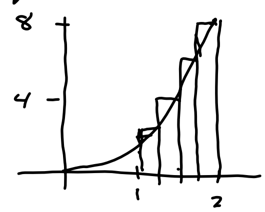
Example: Use Riemann sums to compute

$\int_1^2 x^3\,dx$

Sol: Choose right sums.

$a = 1$ $\quad \Delta x = \frac{b-a}{n} = \frac{1}{n}$

$b = 2$ $\quad x_k = a + k\Delta x = 1 + \frac{k}{n}$



$$\int_1^2 x^3\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left(1+\frac{k}{n}\right)^3 \frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(1+\frac{3k}{n}+\frac{3k^2}{n^2}+\frac{k^3}{n^3}\right)\frac{1}{n}$$

$$= \lim_{n\to\infty} \left[\frac{1}{n}\sum_{k=1}^{n} 1 + \frac{3}{n^2}\sum_{k=1}^{n} k + \frac{3}{n^3}\sum_{k=1}^{n} k^2 + \frac{1}{n^4}\sum_{k=1}^{n} k^3\right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n}\cdot n + \frac{3}{n^2}\frac{n(n+1)}{2} + \frac{3}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2\right]$$

$$= \lim_{n\to\infty} \left[1 + \frac{3}{2}\frac{n(n+1)}{n^2} + \frac{n(n+1)(n+1/2)}{n^3} + \frac{n^2(n+1)^2}{4n^4}\right]$$

$$= 1 + \frac{3}{2} + 1 + \frac{1}{4} = \frac{4+6+4+1}{4}$$

$$= \frac{15}{4}$$

5.2c Definite Integrals

Example: Compute $\int_0^{\pi} \sin x \, dx$ using Riemann Sums

Sol:

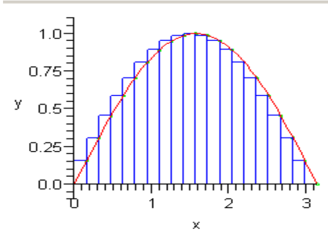
Choose right sums

$a = 0 \qquad \Delta x = \pi/n$

$b = \pi \qquad x_k = 0 + k\frac{\pi}{n}$

$$\int_0^{\pi} \sin x \, dx = \lim_{n\to\infty} \sum_{k=1}^{n} \left[\sin\left(\frac{k\pi}{n}\right)\right]\frac{\pi}{n}$$

$= 2$ Using Maple as shown



$$\sum_{k=1}^{n} \frac{\mathrm{Pi}}{n} \cdot \sin\left(\frac{k \cdot \mathrm{Pi}}{n}\right)$$

$$\frac{1}{2}\,\frac{\pi \sin\left(\frac{\pi}{n}\right)\cos\left(\frac{(n+1)\,\pi}{n}\right)}{n\left(\cos\left(\frac{\pi}{n}\right)-1\right)} - \frac{1}{2}\,\frac{\pi \sin\left(\frac{(n+1)\,\pi}{n}\right)}{n} - \frac{1}{2}\,\frac{\pi \sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)}{n\left(\cos\left(\frac{\pi}{n}\right)-1\right)} + \frac{1}{2}\,\frac{\pi \sin\left(\frac{\pi}{n}\right)}{n} \qquad (1)$$

$$\lim_{n\to\infty}\left(\frac{1}{2}\,\frac{\pi \sin\left(\frac{\pi}{n}\right)\cos\left(\frac{(n+1)\,\pi}{n}\right)}{n\left(\cos\left(\frac{\pi}{n}\right)-1\right)} - \frac{1}{2}\,\frac{\pi \sin\left(\frac{(n+1)\,\pi}{n}\right)}{n} - \frac{1}{2}\,\frac{\pi \sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)}{n\left(\cos\left(\frac{\pi}{n}\right)-1\right)} + \frac{1}{2}\,\frac{\pi \sin\left(\frac{\pi}{n}\right)}{n}\right)$$

$$2 \qquad (2)$$

Properties of Definite Integrals

I. Linearity

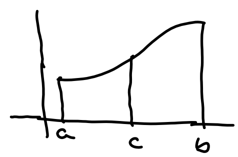
1) $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$

2) $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

II. Other

3) $\int_a^b c\,dx = c[b-a]$

4) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

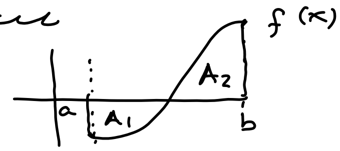


5) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

III Note

6) $\int_a^b f(x)\,dx$ = Area above x-axis − Area below x-axis

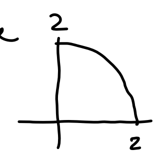
Ex: $\int_a^b f(x)\,dx = A_2 - A_1$



Ex: Evaluate by interpreting in terms of areas

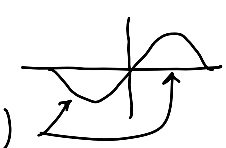
a) $\int_0^2 \sqrt{4-x^2}\,dx$ = Area of $\frac{1}{4}$ circle

$= \frac{\pi \cdot 2^2}{4} = \pi$



b) $\int_{-\pi}^{\pi} \sin x\,dx = 0$

(Areas cancel)



FTC - Fundamental Theorem of Calculus

5.3 Fundamental Theorem of Calculus

Fundamental Theorem of Calculus

Theorem (FTC)

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

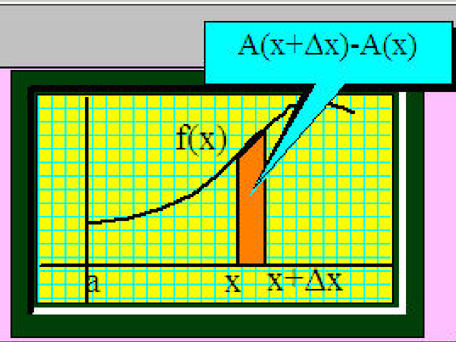
$$\int_a^b f(x)dx = F(b) - F(a)$$

where,

$$F'(x) = f(x)$$

Discussion: Let A(x) denote the area under f(x) in the interval [a,x]. $A(x) = \int_a^x f(t)dt$

$\Delta A = A(x+\Delta x) - A(x) \approx f(x)\,\Delta x$



$$\frac{dA}{dx} = \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = f(x)$$

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Ex. $\frac{d}{dx}\int_2^x \sin t^2 dt = \sin x^2$

Fundamental Theorem of Calculus

Theorem (FTC)

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

$$\int_a^b f(x)dx = F(b) - F(a)$$

where,

$$F'(x) = f(x)$$

(cont) $\int_a^x f(t)dt = F(x) + C$

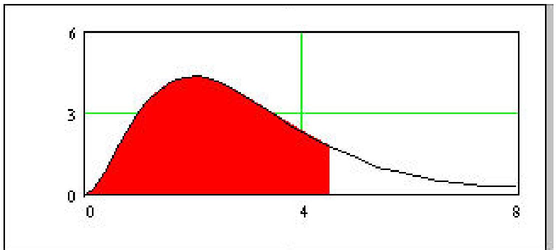
$$\int_a^b f(x)dx = F(b) + C$$

$$\int_a^a f(x)dx = F(a) + C = 0$$

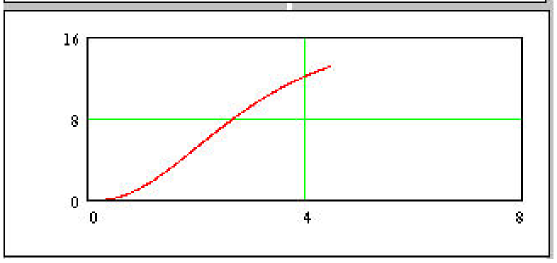
$$C = -F(a)$$

$$\int_a^b f(x)dx = F(b) - F(a)$$

Graphical Interpretation.



$\leftarrow f(t)$



$\leftarrow F(x) = \int_0^x f(t)\,dt$

Examples

$\boxed{7}$ 1. $g(x) = \int_0^x \sqrt{1+2t}\,dt \qquad \frac{d}{dx} g(x) = \sqrt{1+2x}$

$\boxed{10}$ 2 $g(u) = \int_1^u \frac{1}{x+x^2}\,dx \qquad \frac{d}{du} g(u) = \frac{1}{u+u^2}$

$\boxed{12}$ 3) $F(x) = \int_x^{10} \tan\theta\,d\theta \qquad \frac{d}{dx} F(x) = -\tan x$

$\boxed{15}$ 4) $y = \int_3^{\sqrt{x}} \frac{\cos t}{t}\,dt \qquad \frac{d}{dx} y = \frac{\cos\sqrt{x}}{\sqrt{x}} \cdot \frac{d}{dx}(\sqrt{x})$

$= \frac{\cos\sqrt{x}}{2x}$

Chain Rule

$\boxed{18}$ 5) $y = \int_{e^x}^0 \sin^3 t\,dt \qquad \frac{d}{dx} y = \sin^3 e^x \cdot \frac{d}{dx} e^x$

$= e^x \sin^3 e^x$

6) $y = \int_{x^2}^{e^x} \sqrt{t^3+1}\,dt \qquad \frac{d}{dx} y = e^x\sqrt{e^{3x}+1} - 2x\sqrt{x^6+1}$

7) $\frac{d}{dx} \int_{u(x)}^{v(x)} f(t)\,dt = f(v(x))\,v'(x) - f(u(x))\,u'(x)$

5.3a Fundamental Theorem of Calculus

Examples

$\boxed{21}$ 1. $\int_2^8 (4x+3)\,dx = 2x^2+3x\Big]_2^8 = 2[8^2-2^2] + 3[8-2]$
$= 120 - 18 \quad = 102$

$\boxed{24}$ 2. $\int_1^8 \sqrt[3]{x}\,dx = \int_1^8 x^{1/3}\,dx = \frac{3}{4}x^{4/3}\Big|_1^8 = \frac{3}{4}[16-1] = \frac{45}{4}$

$\boxed{28}$ 3. $\int_\pi^{2\pi} \cos\theta\,d\theta = \sin\theta\Big|_\pi^{2\pi} = \sin 2\pi - \sin\pi = 0$

$\boxed{31}$ 4. $\int_0^{\pi/4} \sec^2 t\,dt = \tan t\Big|_0^{\pi/4} = \tan\frac{\pi}{4} - \tan 0 = 1$

$\boxed{38}$ 5. $\int_0^1 \frac{4}{t^2+1}\,dt = \tan^{-1} t\Big|_0^1 = \tan^{-1}1 - \tan^{-1}0 = \frac{\pi}{4}$

$\boxed{40}$ 6. $\int_1^2 \frac{4+u^2}{u^3}\,du = \int_1^2 \left(\frac{4}{u^2} + \frac{1}{u}\right) du$

$= \int_1^2 \left(4u^{-2} + \frac{1}{u}\right) du$

$= -4u^{-1} + \ln|u|\Big|_1^2 = -4\left(\frac{1}{2}-1\right) + \ln 2$

$= 2 + \ln 2$

Indefinite Integrals

6.4 Indefinite Integrals

$\int f(x)\,dx = F(x) + c$ where $F'(x) = f(x)$

The integral of $f(x)$ is the most general antiderivative of $f(x)$

The most basic integrals are given below. The constant of integration has been omitted.

Table of Basic Integration Formulas

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1}\quad (n \neq -1)$	2. $\int \frac{1}{x}\,dx = \ln\lvert x\rvert$
3. $\int e^x\,dx = e^x$	4. $\int a^x\,dx = \frac{a^x}{\ln a}$
5. $\int \sin x\,dx = -\cos x$	6. $\int \cos x\,dx = \sin x$
7. $\int \sec^2 x\,dx = \tan x$	8. $\int \csc^2 x\,dx = -\cot x$
9. $\int \sec x \tan x\,dx = \sec x$	10. $\int \csc x \cot x\,dx = -\csc x$
11. $\int \sec x\,dx = \ln\lvert \sec x + \tan x\rvert$	12. $\int \csc x\,dx = \ln\lvert \csc x - \cot x\rvert$
13. $\int \tan x\,dx = \ln\lvert \sec x\rvert$	14. $\int \cot x\,dx = \ln\lvert \sin x\rvert$
15. $\int \frac{dx}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)$	16. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right)$

Examples:

$\boxed{c}$ 1. $\int \sqrt[3]{x}\,dx = \int x^{1/3}\,dx = \frac{3}{4}x^{4/3} + c = \frac{3}{4}x\sqrt[3]{x} + c$

$\boxed{8}$ 2. $\int x(1+2x^4)\,dx = \int (x + 2x^5)\,dx = \frac{1}{2}x^2 + \frac{1}{3}x^6 + c$

$\boxed{11}$ 3. $\int (2-\sqrt{x})^2\,dx = \int (4 - 4\sqrt{x} + x)\,dx$

$= \int (4 - x^{1/2} + x)\,dx = 4x - \frac{2}{3}x^{3/2} + \frac{x^2}{2} + c$

$= 4x - \frac{2}{3}x\sqrt{x} + \frac{1}{2}x^2 + c$

6.4a Indefinite Integrals.

Examples (Cont.)

$\boxed{10}$ 4. $\int \left(x^2+1+\frac{1}{x^2+1}\right) dx = \frac{1}{3}x^3 + x + \tan^{-1} x + c$

$\boxed{13}$ 5. $\int \frac{\sin x}{1-\sin^2 x} dx = \int \frac{\sin x}{\cos^2 x} dx = \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} dx$

$= \int \tan x \sec x \, dx = \sec x + c$

$\boxed{14}$ 6. $\int \frac{\sin 2x}{\sin x} dx = \int \frac{2 \sin x \cos x}{\sin x} dx$ (Double ∡)

$= \int 2 \cos x \, dx = 2 \sin x + c$

$\boxed{19}$ 7. $\int_{-1}^{0} (2x - e^x) dx = x^2 - e^x \Big|_{-1}^{0} = [-e^0] - [(-1)^2 - e^{-1}] = 1/e$

$\boxed{24}$ 8. $\int_0^9 \sqrt{2t} \, dt = \sqrt{2} \int_0^9 t^{1/2} dt = \sqrt{2} \; \frac{2}{3} t^{3/2} \Big]_0^9 = \sqrt{2} \cdot \frac{2}{3} \cdot 27$

$= 18\sqrt{2}$

10. $\int_1^9 \frac{3x-2}{\sqrt{x}} dx$

$$\lim_{n \to \infty} (3n^2)^{1/n}$$

Let $y = (3x^2)^{1/x}$

$$\ln y = \frac{1}{x} \ln(3x^2)$$

$$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{\left(\frac{1}{3x^2}\right)6x}{1} = 0$$

$$\lim_{x \to \infty} y = e^0 = 1$$

$$\int_{0.5}^{0} e^{x^2} dx \qquad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} \cdots$$

$$\int_0^{1/2} e^{x^2} dx = x + \frac{x^3}{3} + \frac{1}{2!} \frac{x^5}{5} + \frac{1}{3!} \frac{x^7}{7} \cdots \Big]_0^{1/2}$$

$$= \frac{1}{2} + \frac{1}{3 \cdot 8} + \frac{1}{2} \frac{1}{5} \frac{1}{32} + \cdots$$

$$I = \int \frac{t}{1-t^8}\,dt$$

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots \qquad |t| < 1$$

$$\frac{1}{1-t^8} = 1 + t^8 + t^{16} + t^{24} + \cdots$$

$$\frac{t}{1-t^8} = t + t^9 + t^{17} + t^{25} + \cdots$$

$$I = \frac{t^2}{2} + \frac{t^{10}}{10} + \frac{t^{18}}{18} + \cdots \qquad |t| < 1$$

$$\int \tan^{-1} x^2 \, dx$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \qquad |x| < 1$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$$

$$\tan^{-1} x^2 = x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7}$$

$$\int \tan^{-1} x^2 \, dx = \frac{1}{3} x^3 - \frac{1}{3\cdot 7} x^7 + \frac{1}{5\cdot 11} x^{11} + \cdots \qquad |x| < 1$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots \qquad |x| < 1$$

$$x \mapsto \frac{x}{2}$$

$$\frac{1}{1-\frac{x}{2}} = 1 + \frac{x}{2} + \left(\frac{x}{2}\right)^2 + \cdots \qquad \left|\frac{x}{2}\right| < 1$$

$$|x| < 2$$

Sum

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n} (2n)!}$$

$$= \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$e^x = \sum \frac{x^n}{n!}$$

$$\cos x = \sum \frac{(-1)^n x^{2n}}{(2n)!}$$

$\boxed{\#11}$ § 9

$$f(x) = \frac{3}{x^2+x-2} = \frac{3}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$3 = A(x-1) + B(x+2) \qquad = \frac{-1}{2+x} - \frac{1}{1-x}$$

$x=1$ | $3 = 3B$ $\quad B = 1$

$x=-2$ | $3 = -3A$ $\quad A = -1$

$$f(x) = -\frac{1}{2}\left[\frac{1}{1+x/2}\right] - \frac{1}{1-x} \qquad \left|\frac{x}{2}\right| < 1$$

$$= -\frac{1}{2}\left[1 - \frac{x}{2} + \left(\frac{x}{2}\right)^2 - \left(\frac{x}{2}\right)^3 \cdots\right] - \left[1 + x + x^2 + x^3\right.$$

$$= -\frac{3}{2} - \frac{3}{4}x$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x \qquad |x| < \infty$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\frac{1}{2}\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{2\cdot(2n!)}$$