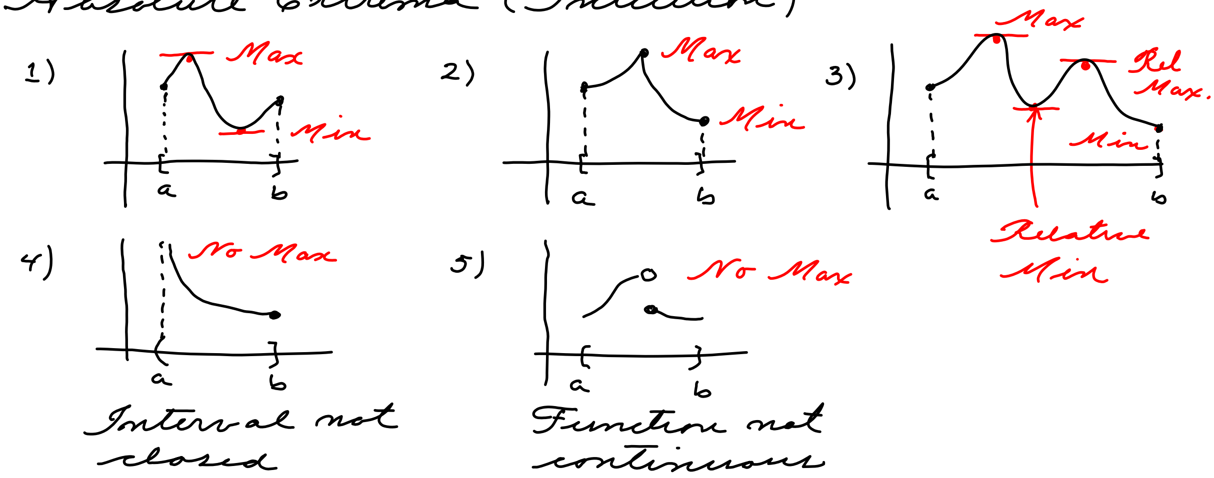
Maximum and Minimum

4.1 Maximum and Minimum

Absolute Extrema (Intuition)



Theorem: A continuous function on a closed interval attains a maximum and a minimum

Theorem: If $f(x)$ has a relative (local) maximum or minimum at a point $x = c$ and $f'(c)$ exists, then $f'(c) = 0$

Definition: A critical point of a function $f(x)$ is a point c where $f'(c)$ or where $f'(c)$ does not exist.

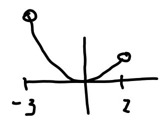
Theorem: If $f(x)$ has a relative extremum at $x = c$ then c is a critical point

Theorem: The extrema of a continuous function on a closed interval occur at critical points or endpoints.

Examples:

Find the absolute and local extrema

1. $f(x) = x^2$ on $(-3, 2)$



Sol: $f'(x) = 2x = 0 \quad x = 0$ c.p

x	f(x)
0	0

$f_{min} = 0$

f_{max} does not exist

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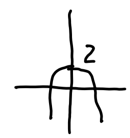
2. $f(x) = 2 - x^4$

$f'(x) = -4x^3 = 0 \quad x = 0$ c.p No endpoints



x	f(x)
0	2

f_{min} does not exist

$f_{max} = 2$

Find the critical points.

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3. $f(t) = 2t^3 + 3t^2 + 6t + 4$

$f'(t) = 6t^2 + 6t + 6 = 0$

$t^2 + t + 1 = 0$

$t = \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1 \pm \sqrt{3}\,i}{2}$

No real solutions

$\therefore$ No c.p.

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4. $g(t) = 5t^{2/3} + t^{5/3}$

$g'(t) = 5 \cdot \frac{2}{3} t^{-1/3} + \frac{5}{3} t^{\frac{2}{3}} = \frac{5}{3}\left[\frac{2}{t^{1/3}} + t^{2/3}\right]$

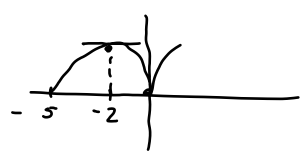
$= \frac{5}{3}\left[\frac{2 + t^{3/3}}{t^{1/3}}\right] = \frac{5}{3}\left[\frac{2+t}{t^{1/3}}\right]$

$g'(t) = 0$ when $t = -2 \qquad t_c = 0, -2$

$g'(t)$ DNE when $t = 0$

Graph: $g(t) = t^{2/3}[5 + t]$

$g(t) = 0 \quad t = 0, -5$



4.1b Maximum and Minimum (Cont.) *Critical Points*.

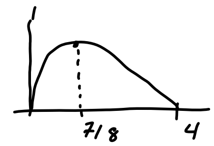
[41]

3. $f(x) = x^{4/5}(x-4)^2$

$$f'(x) = x^{4/5}(2)(x-4) + \frac{4}{5}x^{-\frac{1}{5}}(x-4)^2$$

$$= 2(x-4)\left[x^{\frac{4}{5}} + \frac{2}{5x^{1/5}}(x-4)\right]$$

$$= 2(x-4)\left[\frac{5x^{5/5} + 2(x-4)}{5x^{1/5}}\right] = \frac{2(x-4)(7x-8)}{5x^{1/5}}$$



$f'(x) = 0$ when $x = 4, 7/8$

$f'(x)$ DNE when $x = 0$ $\qquad x_c = 0, \frac{7}{8}, 4$

[43]

4. $f(x) = 2\cos\theta + \sin^2\theta$

$$f'(x) = -2\sin\theta + 2\sin\theta\cos\theta$$

$$= 2\sin\theta(-1 + \cos\theta)$$

$f'(x) = 0$ when $\sin\theta = 0 \qquad \theta = \pm n\pi \leftarrow$ c.p's

$\cos\theta = 1 \qquad \theta = \pm 2n\pi \qquad n = 0, 1, 2, \text{-----}$

[45]

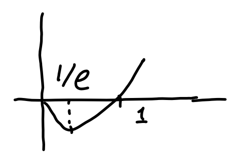
5. $f(x) = x\ln x$

$$f'(x) = x \cdot \frac{1}{x} + \ln x = 1 + \ln x$$

$f'(x) = 0$ when $\ln x = -1 \qquad x = e^{-1} = \frac{1}{e}$

$f'(x)$ DNE if $x = 0$

not in the domain.

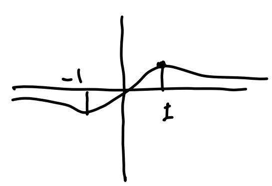


[37]

6. $f(r) = \frac{r}{r^2+1}$

$$f'(r) = \frac{(r^2+1)(r)' - r(r^2+1)'}{(r^2+1)^2}$$

$$= \frac{r^2+1 - r(2r)}{(r^2+1)^2} = \frac{1-r^2}{(r^2+1)^2}$$



$f'(r) = 0$ if $1 - r^2 = 0 \qquad r = \pm 1$

Find the absolute extrema

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1. $f(x) = 2x^3 + 3x^2 + 4$ on $[-2, 1]$

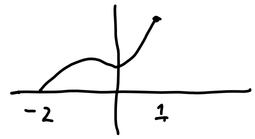
$f'(x) = 6x^2 + 6x$

$= 6x(x+1) = 0 \qquad x = 0, -1$

$f_{max} = 9$

$f_{min} = 0$

x	f(x)
0	4
-1	5
-2	0
1	9



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2. $f(x) = x^2 + \frac{2}{x} \qquad [\frac{1}{2}, 2]$

$f'(x) = 2x - \frac{2}{x^3} = 0$

$x = \frac{1}{x^3}$

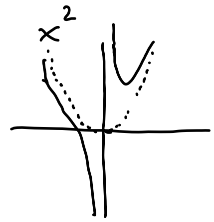
$x^4 = 1$

$x = \pm 1$

$f_{max} = DNE$

$f_{min} = -1$

x	f(x)
1	3
-1	-1
1/2	4.25
2	5



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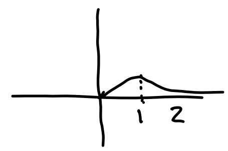
3. $f(x) = xe^{-x} \qquad [0, 2]$

$f'(x) = xe^{-x}(-1) + e^{-x}$

$= e^{-x}(-x+1)$

$f'(x) = 0$ when $x = 1$

x	f(x)	
1	1/e	— f_{max}
0	0	— f_{min}
2	$2/e^2$	



4. $f(x) = \sqrt[3]{x}\,(8 - x)$ on $[0, 8]$

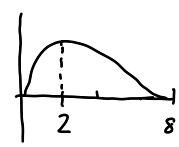
$f'(x) = \frac{1}{3}x^{-2/3}(8 - x) + x^{1/3}(-1)$

$= \frac{8 - x}{3x^{2/3}} - x^{1/3} = \frac{8 - x - 3x^{3/3}}{3x^{2/3}}$

$= \frac{8 - 4x}{3x^{2/3}} = 0 \qquad C.P: \; x = 2, 0$

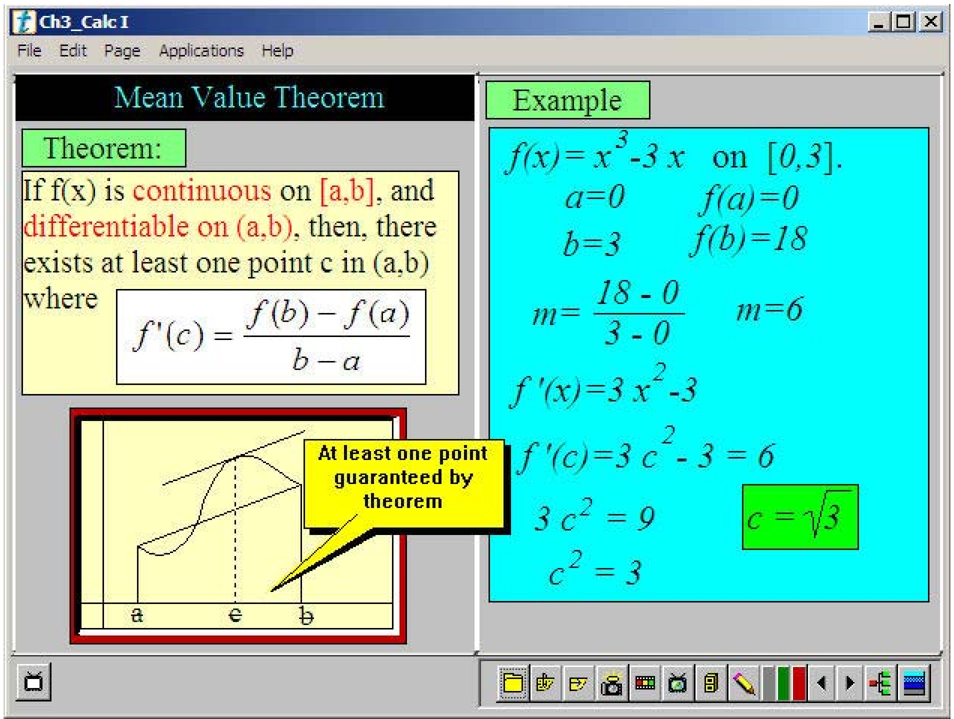
$f_{max} = 6\sqrt[3]{2} \qquad f_{min} = 0$

x	f(x)
0	0
2	$6\sqrt[3]{2}$
8	0



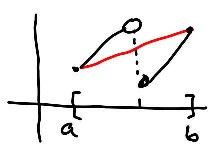
Mean Value Theorem

4.2 MVT



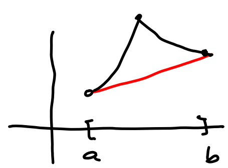
Discussion

1. a)



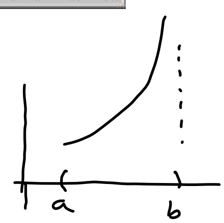
MVT Fails
f not cont.

b)



MVT fails
f not diff.

c)



MVT fails
Int. not closed

2. If $x = f(t)$ on $[a,b]$ represents the position of a particle, then the MVT states that there exists at least one time $t = c$ where the instantaneous velocity $f'(c)$ is equal to the average velocity

$$v_{AVE} = \frac{f(b) - f(a)}{b - a}$$

Examples:
Find all numbers "c" satisfying the conclusion of the MVT

1. $f(x) = x^3 + x - 1$ on $[0, 2]$ $a = 0$ $f(a) = -1$
$f'(x) = 3x^2 + 1$ $b = 2$ $f(b) = 9$
$f'(c) = 3c^2 + 1 = \frac{9+1}{2-0} = 5$

$3c^2 = 4 \quad c^2 = 4/3 \quad c = 2/\sqrt{3}$

2. $f(x) = e^{-2x}$ on $[0, 3]$ $a = 0$ $f(0) = 1$
$f'(x) = -2e^{-2x}$ $b = 3$ $f(3) = e^{-6}$
$f'(c) = -2e^{-2c} = \frac{e^{-6} - 1}{3 - 0}$

$e^{-2c} = \frac{1 - e^{-6}}{6} \qquad -2c = \ln\left(\frac{1 - e^{-6}}{6}\right)$

$c = -\frac{1}{2} \ln\left[\frac{1 - e^{-6}}{6}\right]$

Using the MVT one can prove

A. If $f'(x) = 0$ for all $x \in [a, b]$ then $f(x)$ is constant

B. If $f'(x) = g'(x)$ for all $x \in [a, b]$ then $f(x) = g(x) + c$, where c is constant

19 Ex: Show that $x^2 - 6x + c = 0$ has at most one root in the interval $[-1, 1]$

Sol: Suppose there were two roots a and b. Since $f(a) = f(b) = 0$, by the MVT, there is a point $c \in [-1, 1]$ where $f'(c) = 0$
$f'(x) = x^2 - 6$
$f'(c) = c^2 - 6 = 0 \Rightarrow c = \pm\sqrt{6} \notin [-1, 1]$. This is a contradiction so the assumption must be wrong. No such roots a and b exist.

Derivatives and Shapes of Graphs

Sunday, March 12, 2006
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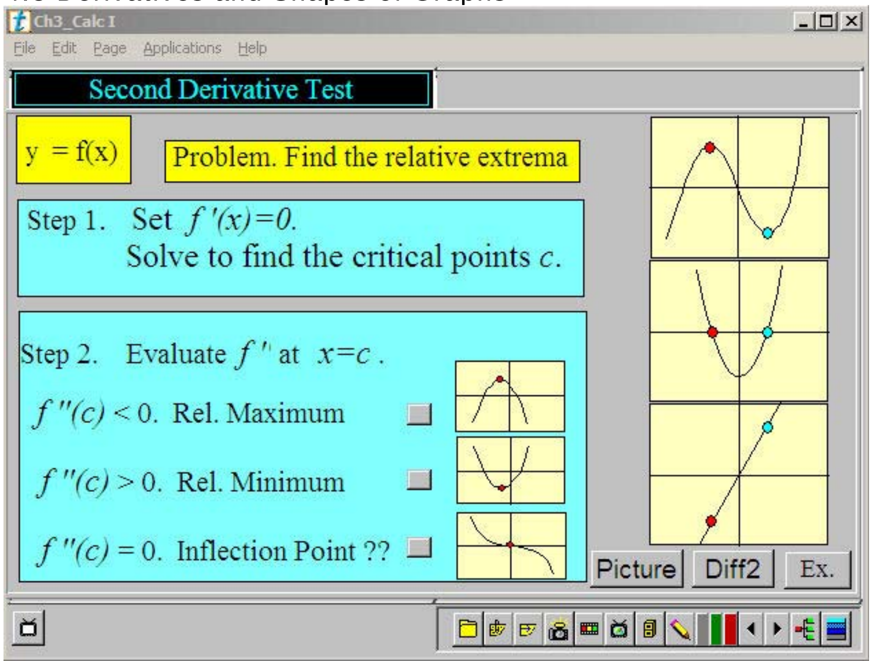
4.3 Derivatives and Shapes of Graphs

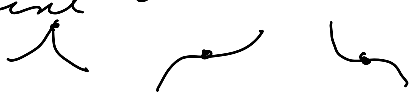


First Derivative

$f'(c)$ $\begin{cases} > 0 & \text{Increasing} \\ < 0 & \text{Decreasing} \\ = 0 & \text{Horizontal} \end{cases}$

$f''(c)$ $\begin{cases} > 0 & \text{Concave up} \\ < 0 & \text{Concave down} \\ = 0\ ? & \text{Can't tell (IP ?)} \end{cases}$

Definition: A point c where f(x) changes concavity is called an inflection point

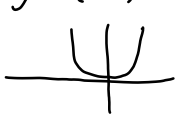


Fact: If $x = c$ is an inflection point and f'' exists at $x = c$ then $f''(c) = 0$

- The converse is false!

Examples

1. $f(x) = x^4$
   $f''(0) = 0$

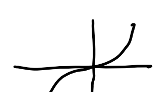


loc Min

2. $f(x) = -x^4$
   $f''(0) = 0$

loc Max

3. $f(x) = x^3$
   $f''(0) = 0$



I. P.

Examples

1. $f(x) = x^3 - 3x = 0 \Rightarrow x = 0, \pm\sqrt{3}$ Roots

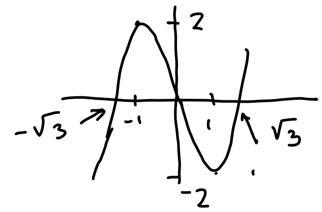
$f'(x) = 3x^2 - 3 = 0 \Rightarrow x^2 - 1 = 0, \quad x = \pm 1$ C.P

$f''(x) = 6x = 0 \Rightarrow x = 0$ possible I.P.

$f''(1) > 0$ local min at $x = 1$

$f''(-1) < 0$ local max at $x = -1$

at $x = 0$ Inflection Point



4.4 L'Hopital's Theorem.

Let $f(x)$ and $g(x)$ be differentiable functions at $x=a$. If $f(a)$ and $g(a)$ are both 0, then

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x=a$. The substitution results in an expression of the form "0/0". Such an expression is an example of a so-called "indeterminate form."

L'Hopital's theorem provides an easy method to evaluate such limits

Theorem: Suppose that $f(a)=g(a)=0$ and that $f'(a)$ and $g'(a)$ both exist. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Similarly, if $f(x)$ and $g(x)$ approach ∞ as $x\to\infty$ and $f(x)$ & $g(x)$ are differentiable, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

L'Hopital's theorem allows us to evaluate limits involving indeterminate forms of types:

$$\frac{0}{0},\ \frac{\infty}{\infty},\ \infty\cdot 0,\ \infty-\infty,\ 1^{\infty},\ 0^{0},\ \infty^{0}$$

provided one can manipulate the expressions algebraically to be of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$!

Examples

1. $\lim_{x\to 2} \frac{x^2-4}{x-2} \overset{L.H}{=} \lim_{x\to 2} \frac{2x}{1} = 2$

2. $\lim_{x\to 1} \frac{x^{100}-1}{x^2-1} \overset{L.H}{=} \lim_{x\to 1} \frac{100x^{99}}{2x} = 50$

3. $\lim_{x\to 0} \frac{\sin 5x}{3x} \overset{L.H}{=} \lim_{x\to 0} \frac{5\cos 5x}{3} = \frac{5}{3}$

4. $\lim_{x\to 0} \frac{e^x-1-x}{x^2} \overset{L.H}{=} \lim_{x\to 0} \frac{e^x-1}{2x} \overset{L.H}{=} \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$

5. $\lim_{x\to\infty} \frac{x^4-3}{3x^4+5x} \overset{L.H}{=} \lim_{x\to\infty} \frac{4x^3}{12x^3+5} \overset{L.H}{=} \lim_{x\to\infty} \frac{12x^2}{36x^2} = \frac{1}{3}$

6. $\lim_{x\to\infty} \frac{\ln x}{x} \overset{L.H}{=} \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$

7. $\lim_{x\to\infty} x\sin(1/x) = \lim_{x\to\infty} \frac{\sin(1/x)}{(1/x)}$ "$\frac{0}{0}$"

"$\infty \cdot 0$"

$\overset{L.H}{=} \lim_{x\to\infty} \frac{(-1/x^2)\cos(1/x)}{(-1/x^2)}$

$= \lim_{x\to\infty} \cos(1/x) = \cos 0 = 1$

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8. $\lim_{x\to 1^+} (\ln x)(\tan \frac{\pi}{2}x) = \lim_{x\to 1^+} \frac{\ln x}{\cot\frac{\pi}{2}x}$ "$\frac{0}{0}$"

"$0 \cdot \infty$"

$= \lim_{x\to 1^+} \frac{1/x}{\frac{\pi}{2}\csc^2\frac{\pi}{2}x}$

$= \lim_{x\to 1^+} \frac{\sin^2\frac{\pi}{2}x}{\frac{\pi}{2}x} = \frac{2}{\pi}$

9. $\lim_{x\to\infty} \frac{x^n}{e^x} \overset{L.H}{=} \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} = \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x} = \cdots$

Apply L.H n times

$= \lim_{x\to\infty} \frac{n!}{e^x} = 0$

Examples

10. $\lim_{x\to\infty}(x-\ln x) = \lim_{x\to\infty} x\left(1-\frac{\ln x}{x}\right) = \infty$ "$\infty - \infty$" "$\infty \cdot (1-0)$"

11. $\lim_{x\to 0^+}\left(\frac{1}{\sin x}-\frac{1}{x}\right) = \lim_{x\to 0^+}\frac{x-\sin x}{x\sin x}$ "$\infty - \infty$" "$0/0$"

$\overset{L.H}{=} \lim_{x\to 0^+}\frac{1-\cos x}{\sin x + x\cos x}$ "$\frac{0}{0}$"

$\overset{L.H}{=} \lim_{x\to 0^+}\frac{\sin x}{\cos x + \cos x - x\sin x} = \frac{0}{2} = 0$

12. $\lim_{x\to\infty}\frac{3x}{\sqrt{x^2+4}} = \lim_{x\to\infty}\sqrt{\frac{9x^2}{x^2+4}}$

$= \sqrt{\lim_{x\to\infty}\frac{9x^2}{x^2+4}} \overset{LH}{=} \sqrt{\lim_{x\to\infty}\frac{18x}{2x}} = \frac{3}{\sqrt{2}}$

13. $\lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right) = \lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right)\frac{\left(x+\sqrt{x^2+x}\right)}{\left(x+\sqrt{x^2+x}\right)}$ "$\infty - \infty$"

$= \lim_{x\to\infty}\frac{x^2-(x^2+x)}{x+\sqrt{x^2+x}}$

$= \lim_{x\to\infty}\frac{x}{x+\sqrt{x^2+x}}$ $\quad \frac{(1/x)}{(1/x)}$

$= \lim_{x\to\infty}\frac{1}{1+\sqrt{\frac{x^2+x}{x^2}}}$

$= \frac{1}{1+\sqrt{\lim_{x\to\infty}\frac{x^2+x}{x^2}}}$

$\overset{L.H}{=} \frac{1}{1+\sqrt{\lim_{x\to\infty}\frac{2x}{2x}}} = \frac{1}{1+1} = \frac{1}{2}$

Logarithmic L'Hopital's rule

Use to evaluate limits involving $[f(x)]^{g(x)}$

Examples

Recall: $\ln y = z \Leftrightarrow y = e^z$

1. $\lim_{x\to\infty} x^{1/x}$ "∞^0"

Let $y = x^{1/x}$ $\quad \ln y = \frac{1}{x}\ln x$ "$\frac{\infty}{\infty}$"

$$\lim_{x\to\infty} \ln y \overset{L.H}{=} \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

Since $\ln y \to 0$ $\quad y \to 1$

2. <u>Euler's Limit</u>

$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$ "1^∞" Let $y = \left(1+\frac{1}{x}\right)^x$

$\ln y = x \ln\left(1+\frac{1}{x}\right) = \frac{\ln\left(1+\frac{1}{x}\right)}{1/x}$ "$\frac{0}{0}$"

$$\lim_{x\to\infty} \ln y \overset{L.H}{=} \lim_{x\to\infty} \frac{\left(\frac{1}{1+1/x}\right)\left(-1/x^2\right)}{\left(-1/x^2\right)}$$

$$= \lim_{x\to\infty} \frac{1}{1+1/x} = 1$$

Since $\ln y \to 1$, $\quad y \to e^1 = e$

So: $\boxed{\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e}$

This limit is the basis for the notion of compound interest. It states that \$1 continuously compounded at a rate of 1% per year yields e = 2.71. dollars after 1 year.

Curve Sketching

4.5 Curve Sketching--Guidelines: $y=f(x)$

1. Domain:
   a. Figure out any points or regions where the function is not defined.
   b. If the function has denominators with factors of the form (x-c), then $x=c$ is a vertical asymptote
2. Intercepts:
   a. y-intercept - Set $x=0$ and evaluate y.
   b. x-intercept - Set $y=0$ and solve for x
3. Symmetry -Even/Odd:
   a. Is $f(-x) = f(x)$? If so, $f(x)$ is symmetric with respect to the y-axis.
      This happens in particular if the function contains only even powers of x, cos(x) or cosh(x).
   b. Is $f(-x) = -f(x)$? If so, the function is symmetric with respect to the origin. This happens in particular if the function contains only odd powers of x, or sin(x), or sinh(x)
4. Asymptotes:
   a. Use L'Hospital's theorem as necessary to evaluate the limit as x goes to infinity to find the horizontal asymptotes.
   b. Use L'Hospital's theorem as necessary to evaluate the limit as x goes to c, for any point c where $f(c)$ is an undetermined value.
5. Second Derivative Test
   a. Use the first derivative to find all critical points.
   b. Use the second derivative test to find all relative extrema.
   c. If $f''(c) = 0$, investigate if $x=0$ is an inflection point.

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Solution Take the first derivative and set equal to 0

$f(x) = \dfrac{1}{x^2 + 4}$

$f'(x) = \dfrac{-2x}{(x^2+4)^2} = 0$

Critical Point

$x = 0$

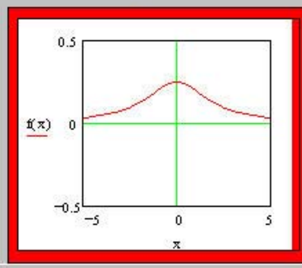
$f''(x) = \dfrac{(x^2+4)^2(-2) + (2x)\,2(x^2+4)(2x)}{(x^2+4)^4}$

$f''(x) = \dfrac{2(x^2+4)[-(x^2+4) + 4x^2]}{(x^2+4)^4}$

$f''(x) = \dfrac{2(3x^2-4)}{(x^2+4)^3}$

$f''(0) < 0$

Rel. Max. at $x = 0$



Continue

Inflection pts.

$f''(x) = 0$

when

$3x^2 - 4 = 0$

$3x^2 = 4$

$x = \pm \dfrac{2}{\sqrt{3}}$

Ch3_Calc I

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Solution Take the first derivative and set equal to 0

$f(x) = \dfrac{1}{x^2 - 4}$

$f'(x) = \dfrac{-2x}{(x^2-4)^2} = 0$

Critical Point

$x = 0$

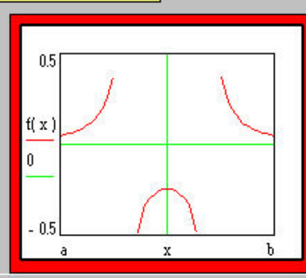
$f''(x) = \dfrac{(x^2-4)^2(-2) + (2x)\,2(x^2-4)(2x)}{(x^2-4)^4}$

$f''(x) = \dfrac{2(x^2-4)[-(x^2-4) + 4x^2]}{(x^2-4)^4}$

$f''(x) = \dfrac{2(3x^2+4)}{(x^2-4)^3}$

$f''(0) < 0$

Rel. Max. at $x = 0$



$f'' \neq 0$

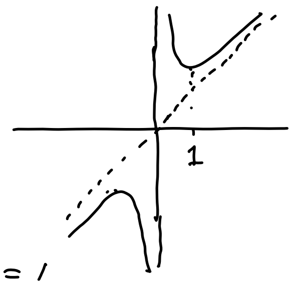
No IP's

Examples:

1. $f(x) = x + \frac{1}{x}$

a) Odd function

b) Asymp: $x = 0 \quad y = x$



$f'(x) = 1 - \frac{1}{x^2} = 0 \Rightarrow x = \pm 1$ C.P

$f''(x) = \frac{2}{x^3}$

$f''(1) > 0$ ∪ loc min at $x = 1$

$f''(-1) < 0$ ∩ loc max at $x = -1$

$f''(x) \neq 0$ No I.P.

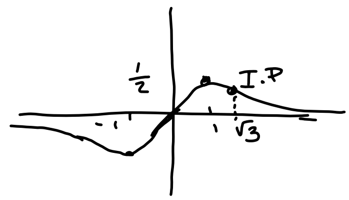
2. $f(x) = \frac{x}{x^2+1}$

a) Odd function

b) as $x \to \infty \quad y \to 0$

$y = 0$ Asymptote

c) $f(0) = 0$



$f'(x) = \frac{(x^2+1)1 - x(2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = 0 \Rightarrow x = \pm 1$

$f''(x) = \frac{(x^2+1)^2(-2x) - (1-x^2)2(x^2+1)2x}{(x^2+1)^4}$

$= \frac{2x(x^2+1)[-(x^2+1) - 2(1-x^2)]}{(x^2+1)^4}$

$= \frac{2x(x^2-3)}{(x^2+1)^3} = 0 \Rightarrow x = 0, \pm\sqrt{3} \leftarrow$ Possible I.P

$f''(1) < 0$ loc max at $x = 1$

$f''(-1) > 0$ loc min at $x = -1$

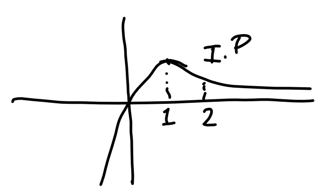
$\Rightarrow$ I.P at $x = 0, \pm\sqrt{3}$

3. $f(x) = xe^{-x}$

a) $f(0) = 0$

b) as $x \to \infty$, $f(x) \to 0$

$y = 0$ Asymptote



$f'(x) = e^{-x} - xe^{-x}$

$= e^{-x}(1-x) = 0 \quad x = 1$ C.P.

$f''(x) = e^{-x}(-1) - (1-x)e^{-x}$

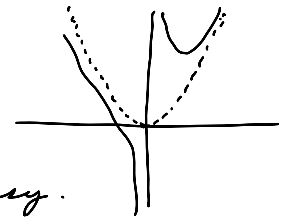
$= e^{-x}(x-2)$

$f''(1) < 0$ loc max at $x = 1$

$f''(x) = 0$ at $x = 2$ I.P.

4.5c Curve Sketching.(Cont)

Example



4. $f(x) = \frac{x^3+1}{x}$

a) $x=0$ asy.

b) $f(x) = x^2 + \frac{1}{x}$

as $x \to \infty$

$y \to x^2$ $\quad y = x^2$ asy.

$f(x) = x^2 + \frac{1}{x}$

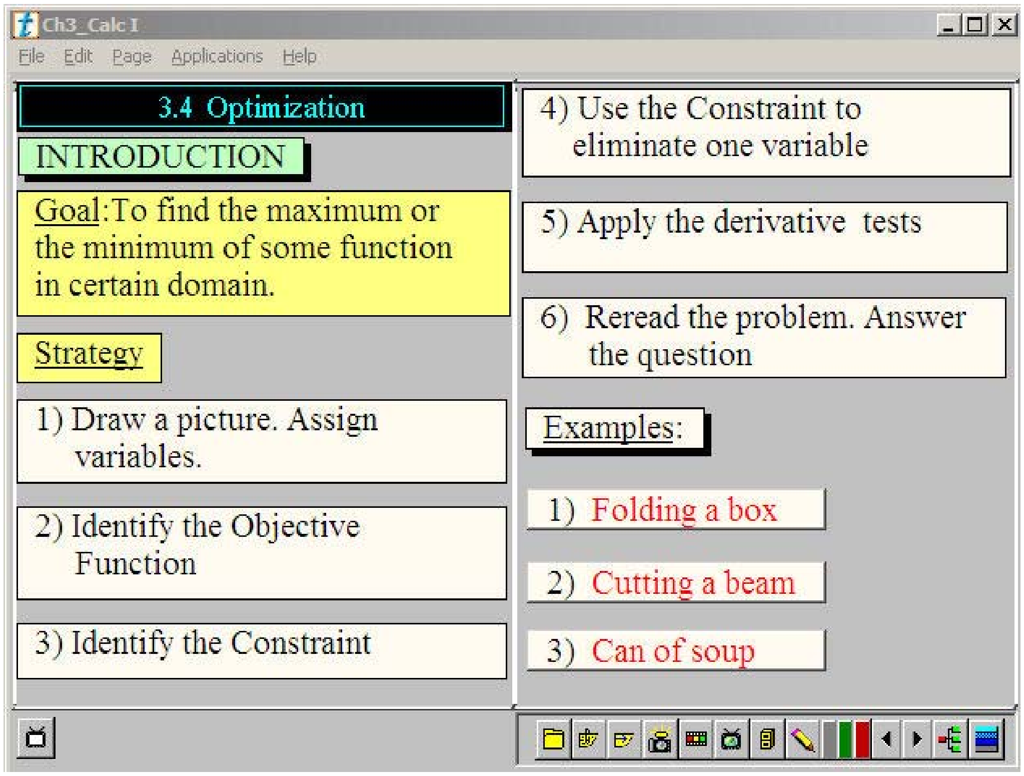
$f'(x) = 2x - \frac{1}{x^2} = 0 \qquad 2x = \frac{1}{x^2} \Rightarrow x^3 = \frac{1}{2} \qquad x = \frac{1}{\sqrt[3]{2}}$ c.p

$f''(x) = 2 + \frac{2}{x^3} \qquad f''\left(\frac{1}{\sqrt[3]{2}}\right) > 0$ loc min at $x = \frac{1}{\sqrt[3]{2}}$

$f''(x) = 0$ if $-2 = \frac{2}{x^3} \Rightarrow x^3 = -1$ I.P @ $x = -1$

Optimization

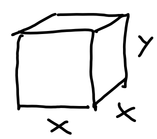
4.7 Optimization



10 **Example:**

1. A box with a square base and open top must have a volume of 32,000 cm^3. Find the dimensions that minimize the amount of material used

Sol



$V = x^2 y = 32000 \Rightarrow y = 32000/x^2$

$A = x^2 + 4xy$

$= x^2 + 4x\left(\frac{32000}{x^2}\right) = x^2 + \frac{128000}{x}$

$A' = 2x - \frac{128000}{x^2} = 0$

$2x = \frac{128000}{x^2}$

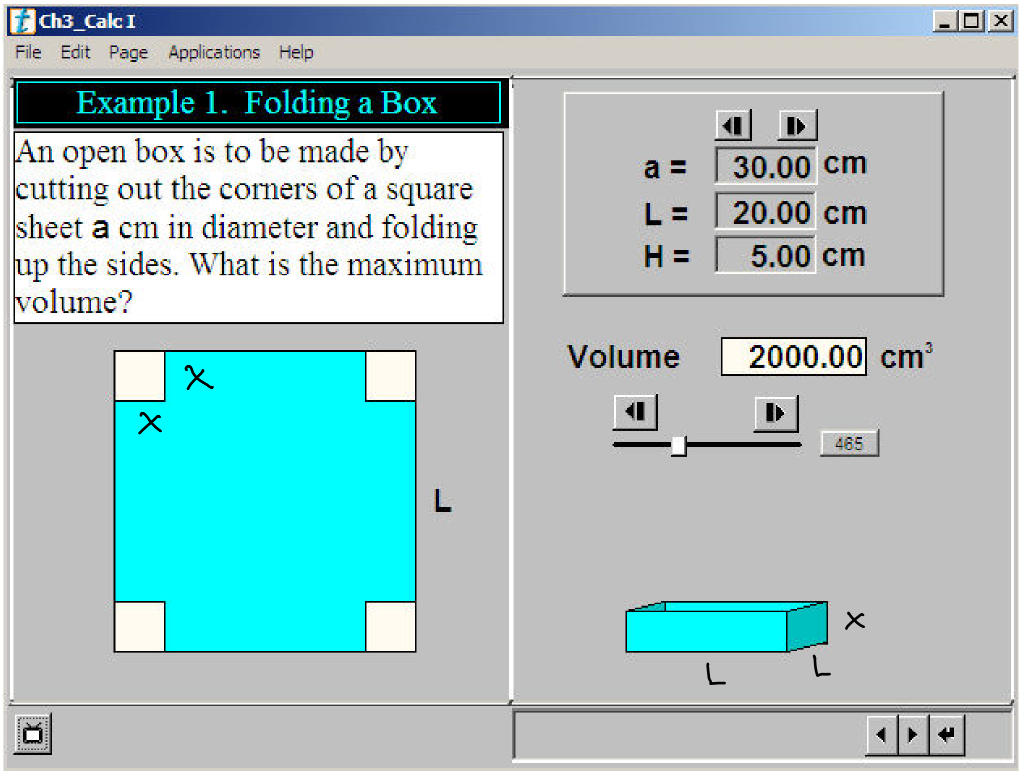
$x^3 = 64000$

$x = 40$

$A'' = 2 + \frac{256000}{x^3}$

$A''(40) > 0$ loc min @ $x = 40,\ y = 20$

Ans: $40 \times 40 \times 20$



Say: $a = 30$ cm $\Rightarrow$ $L = 30 - 2x$

$V = LWH = x(30-2x)^2 \qquad 0 \le x \le 15$

$V' = (30-2x)^2 + x(2)(30-2x)(-2)$

$= (30-2x)[30-2x-4x]$

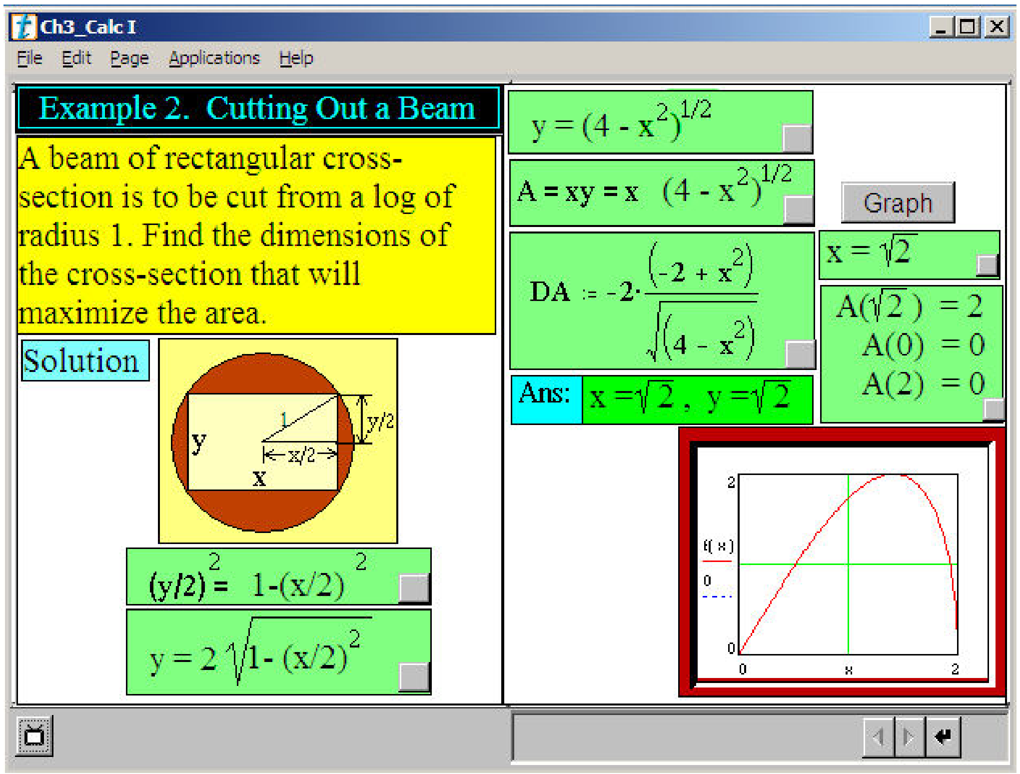
$= 2(15-x)(30-6x) = 0$

$x = 15 \qquad 6x = 30$

$x = 5$ C.P.

Closed Interval $\Rightarrow$ Check C.P, endpts

x	V
0	0
15	0
5	$5(20)^2 = 2000 = V_{MAX}$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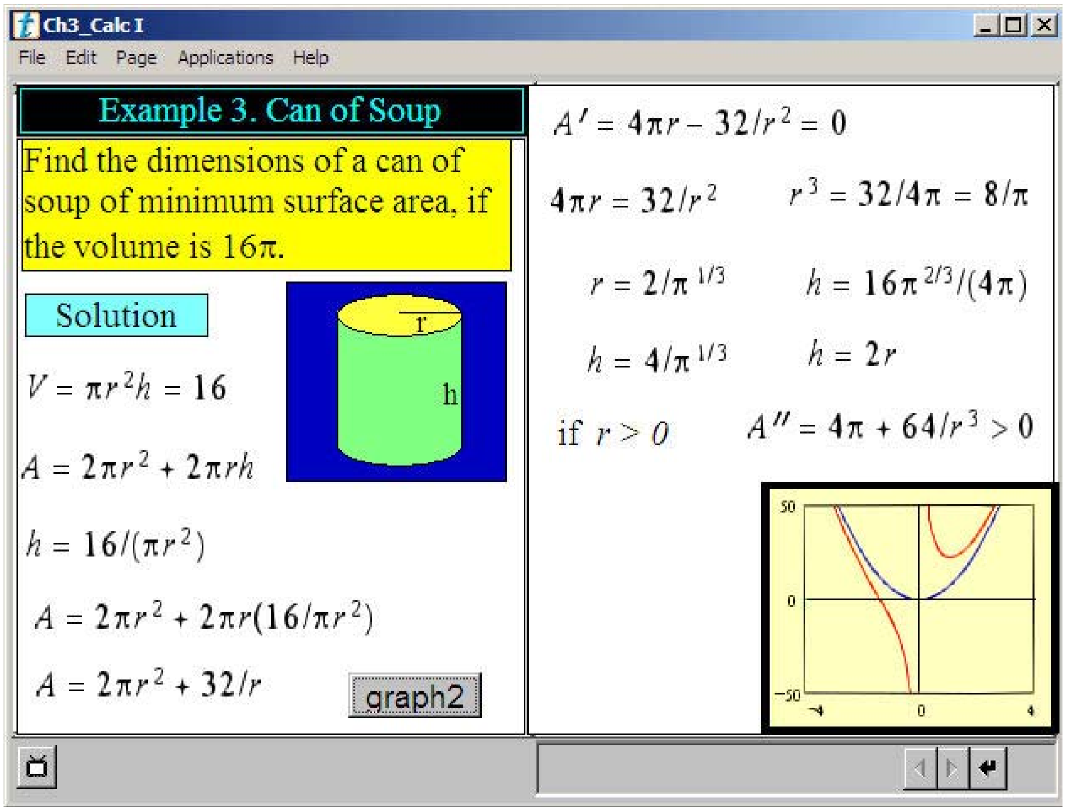
Example 2. Cutting Out a Beam
A beam of rectangular cross-section is to be cut from a log of radius 1. Find the dimensions of the cross-section that will maximize the area.
Solution
y
x
1
x/2
y/2
$(y/2)^2 = 1-(x/2)^2$
$y = 2\sqrt{1-(x/2)^2}$
$y = (4 - x^2)^{1/2}$
$A = xy = x\,(4 - x^2)^{1/2}$
Graph
$DA := -2\cdot\frac{(-2 + x^2)}{\sqrt{(4 - x^2)}}$
$x = \sqrt{2}$
$A(\sqrt{2}) = 2$
$A(0) = 0$
$A(2) = 0$
Ans: $x = \sqrt{2}$, $y = \sqrt{2}$
f(x)
2
0
0
x
2

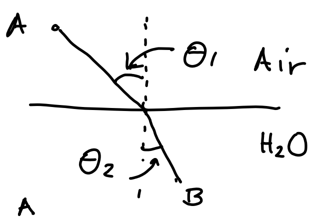
25



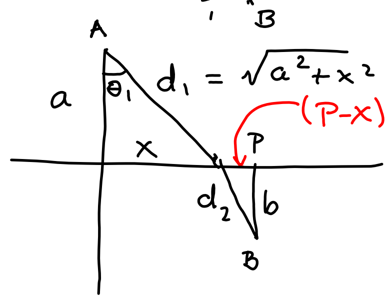
51 **Example:** Let v_1 = velocity of light in air
v_2 = velocity of light in water

Fermat's principle states light will travel from A to B (see fig) so as to minimize the time. Show that

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2} \quad \text{(Snell's Law)}$$



Sol: From the picture



$$d_1 = \sqrt{a^2+x^2} \qquad d_2 = \sqrt{(P-x)^2+b^2}$$

$$t = \frac{d_1}{v_1} + \frac{d_2}{v_2}$$

$$t = \frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{(P-x)^2+b^2}}{v_2}$$

$$\frac{dt}{dx} = \frac{x}{v_1\sqrt{a^2+x^2}} + \frac{-(P-x)}{v_2\sqrt{(P-x)^2+b^2}} = 0$$

$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0$$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

Left to the reader
Check endpoints
$x = 0$, $x = P$.

$$\frac{dt}{dx} = \frac{x}{v_1\sqrt{a^2+x^2}} + \frac{-(P-x)}{v_2\sqrt{(P-x)^2+b^2}} = 0$$

$$\frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} = 0$$

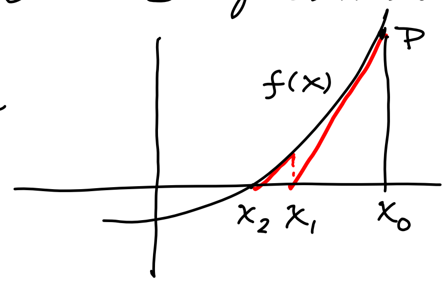
$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

Left to the reader
Check endpoints
$x=0$, $x=P$.

Newton's Method

4.9 Newton's Method

Newton's method is an efficient procedure to find the roots of smooth functions.



Let x_0 be an initial guess to approximate a root of $f(x)$

- Find the equation of the tangent to $f(x)$ at x_0
- Use the point of intersection of the tangent line with the x-axis as the next approximation x_1
- Repeat the procedure

$P(x_0, f(x_0))$ Slope: $f'(x_0)$

Equation of tangent ← Set $y = 0$, $x = x_1$

$$y - f(x_0) = f'(x_0)(x - x_0)$$

$$-f(x_0) = f'(x_0)(x_1 - x_0)$$

$$-\frac{f(x_0)}{f'(x_0)} = x_1 - x_0 \Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Iterate the procedure $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

Newton's method is best implemented in Maple.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Newton's Method.

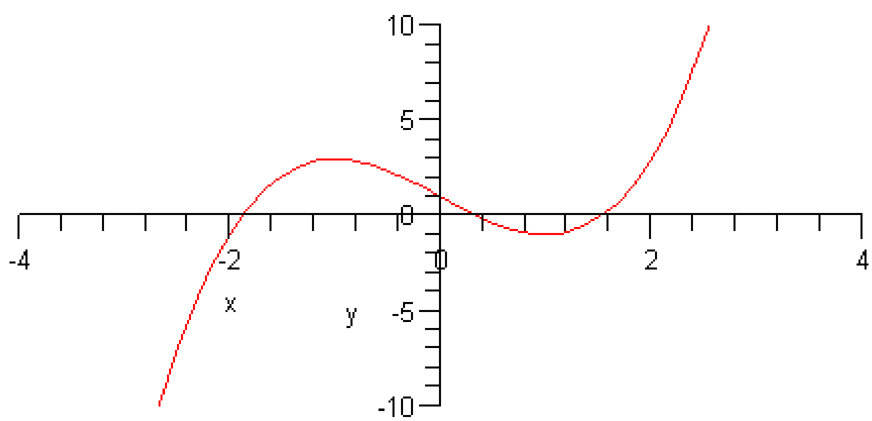
unassign(xn);

$f := x \rightarrow x^3 - 3 \cdot x + 1;$

$$x \rightarrow x^3 - 3x + 1 \quad (1)$$

$$x \rightarrow x^3 - 3x + 1 \quad (2)$$

$plot(f(x),\ x = -4..4,\ y = -10..10);$



$df := D(f);$

$$x \rightarrow 3x^2 - 3 \quad (3)$$

$x[0] := 2;$

$$2 \quad (4)$$

for n **from** 0 **to** 5 **do** $x[n+1] := evalf\left(x[n] - \frac{f(x[n])}{df(x[n])}\right)$ **od**

$$1.532088886 \quad (5)$$

4.10 Antiderivatives

Definition: $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.

Any two antiderivatives of $f(x)$ differ at most by a constant. So, if $F(x)$ is an antiderivative of $f(x)$, the most general antiderivative is

$F(x) + C$. C - arbitrary const.

Examples: Find the most general antiderivative of the given functions

1. $f(x) = 3x^2 + 5x - 2$ $\quad F(x) = x^3 + \frac{5}{2}x^2 - 2x + C$

2. $f(x) = 6\sqrt{x} - \sqrt[6]{x}$ $\quad F(x) = 6 \cdot \frac{2}{3}x^{3/2} - \frac{6}{7}x^{7/6} + C$

$$= 4x\sqrt{x} - \frac{6}{7}x\sqrt[6]{x} + C$$

3. $f(x) = 4e^x - \sec^2 x$ $\quad F(x) = 4e^x - \tan x + C$

4. $f(\theta) = \frac{\sin\theta}{\cos^2\theta}$ $\quad f(x) = \frac{\sin\theta}{\cos\theta} \cdot \frac{1}{\cos\theta} = \tan\theta \sec\theta$

$$F(x) = \sec\theta + C$$

To evaluate the constant C one must be given conditions.

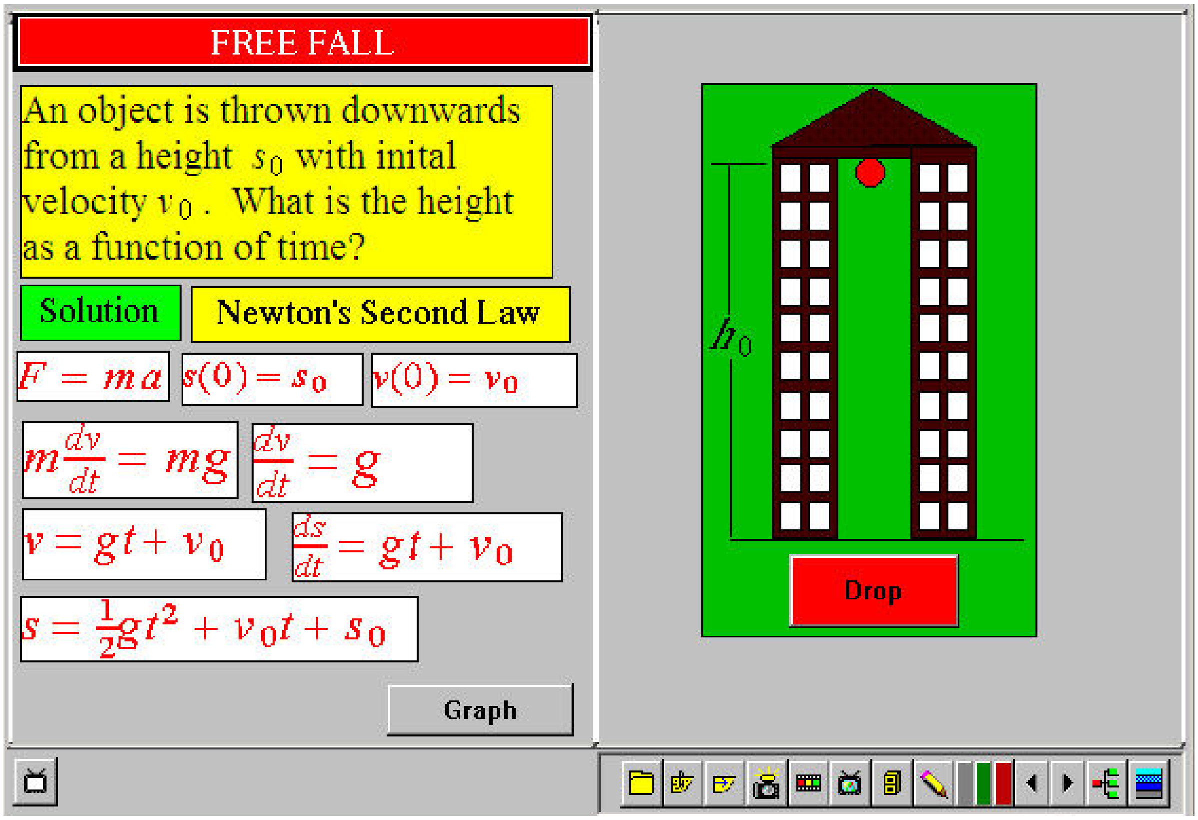
Ex Find $f(x)$

5. $f'(x) = 1 - 6x$, $f(0) = 8$ $\quad f(x) = x - 3x^2 + C$

$f(0) = C = 8 \Rightarrow f(x) = x - 3x^2 + 8$

6. $f''(x) = 2 - 12x$, $f(0) = 3$, $f'(0) = 2$ $\quad f'(x) = 2x - 6x^2 + C = 2x - 6x^2 + 2$

$$f(x) = x^2 - 2x^3 + 2x + D$$
$$= x^2 - 2x^3 + 2x + 3$$



If one follows the convention that g is positive ($g = 9.8\ m/s^2$) but gravity points down, then the free fall equations become

$$v = -gt + v_0$$
$$s = -\frac{1}{2}gt^2 + v_0 t + s_0$$

In USA units
$g = 32\ ft/s^2$

Example: A stone was dropped off a cliff and hits the ground with a speed of 120 ft/s. What was the height of the cliff?

$v_0 = 0, \quad v = -gt + v_0$

$= -gt$

$s = -\frac{1}{2} g t^2 + s_0$

$-120 = -16t$

$t = \frac{120}{16} = \frac{15}{2}$

$0 = -\frac{1}{2} 16 \left(\frac{15}{2}\right)^2 + s_0$

$s_0 = \frac{16}{8}(15)^2 = 450 \text{ ft}$