

Polynomials and Exponentials

3.1 Polynomials and Exponentials

We would like to get formulas for the derivative of x^n .

Recall: $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

$n=0$ $f(x) = x^0 = 1$
 $f'(x) = 0$

$n=1$ $f(x) = x^1 = x$
 $f'(x) = 1$ (slope = 1)

$n=2$ $f(x) = x^2$
 $f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$
 $= \lim_{h \rightarrow 0} \frac{1}{h} [x^2 + 2hx + h^2 - x^2]$
 $= \lim_{h \rightarrow 0} \frac{1}{h} [2hx + h^2] = \lim_{h \rightarrow 0} [2x + h]$
 $= 2x$

$n=3$ $f(x) = x^3$
 $f'(x) = \lim_{h \rightarrow 0} \frac{1}{h} [(x+h)^3 - x^3]$
 $= \lim_{h \rightarrow 0} \frac{1}{h} [x^3 + 3x^2h + 3xh^2 + h^3 - x^3]$
 $= \lim_{h \rightarrow 0} \frac{1}{h} [3x^2h + 3xh^2 + h^3]$
 $= \lim_{h \rightarrow 0} [3x^2 + 3xh + h^2]$
 $= 3x^2$

Summary

$f(x)$	$f'(x)$
x^0	0
x^1	1
x^2	$2x$
x^3	$3x^2$

3.1a Polynomials (cont.)

$$f(x) = x^n$$

First, we need to review Newton's binomial formula

$$(x+h)^1 = 1x + 1h$$

$$(x+h)^2 = 1x^2 + 2xh + 1h^2$$

$$(x+h)^3 = 1x^3 + 3x^2h + 3xh^2 + 1h^3$$

n	Coeff			
1	1	1		
2	1	2	1	
3	1	3	3	1

- Pattern:
1. Powers of x decrease by 1
 2. Powers of h increase by 1
 3. Coefficients given by Pascal's Triangle

Formula for the coefficients

Let n = power of the binomial

k = power of x in the expansion

$\binom{n}{k}$ = Coefficient of x^k

$$\text{Then } \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$n! = n(n-1)(n-2) \dots 2 \cdot 1$$

$$0! = 1$$

Example $n=3$

$$\binom{3}{0} = \frac{3!}{0!(3-0)!} = 1$$

$$\binom{3}{1} = \frac{3!}{1!(3-1)!} = \frac{3!}{2!} = \frac{3 \cdot 2 \cdot 1}{2 \cdot 1} = 3$$

$$\binom{3}{2} = \frac{3!}{2!(3-2)!} = \frac{3!}{2!} = \frac{3 \cdot 2 \cdot 1}{2 \cdot 1} = 3$$

$$\binom{3}{3} = \frac{3!}{3!(3-3)!} = \frac{3!}{3!0!} = \frac{3!}{3!} = 1$$

$$\text{So: } (x+h)^3 = \binom{3}{0}x^3h^0 + \binom{3}{1}x^2h^1 + \binom{3}{2}x^1h^2 + \binom{3}{3}x^0h^3$$

3.1b Polynomials (Cont.)

$$(x+h)^n = \binom{n}{0} x^n h^0 + \binom{n}{1} x^{n-1} h^1 + \binom{n}{2} x^{n-2} h^2 + \binom{n}{3} x^{n-3} h^3 + \dots$$

$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{n!} = 1$$

$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{n!}{(n-1)!} = \frac{n(n-1)(n-2)\dots 2 \cdot 1}{(n-1)(n-2)\dots 2 \cdot 1} = n$$

$$\therefore (x+h)^n = x^n + n x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \dots$$

$$(x+h)^n - x^n = n x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \binom{n}{3} x^{n-3} h^3 + \dots$$

$$\frac{(x+h)^n - x^n}{h} = n x^{n-1} + \binom{n}{2} x^{n-2} h + \binom{n}{3} x^{n-3} h^2 + \dots$$

$$\begin{aligned} \frac{d}{dx} x^n &= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h \rightarrow 0} [n x^{n-1} + \binom{n}{2} x^{n-2} h + \binom{n}{3} x^{n-3} h^2 + \dots] \\ &= n x^{n-1} \end{aligned}$$

Formula: $\boxed{\frac{d}{dx} (x^n) = n x^{n-1}}$ Power Rule.

Examples

$$1) \frac{d}{dx} x^5 = 5x^4 \quad 3) \frac{d}{dx} 3x^7 = 3 \frac{d}{dx} x^7 = 3(7x^6) = 21x^6$$

$$2) \frac{d}{dx} x^{10} = 10x^9 \quad 4) \frac{d}{dx} (x^4 + x^2) = 4x^3 + 2x$$

$$5) \frac{d}{dx} [3x^5 + 4x^2 + 5] = 5(3)x^4 + 2(4)x + 0 = 15x^4 + 8x$$

Note. The power rule is true for any n , not just integers!

$$\text{Ex } \frac{d}{dx} \left(\frac{1}{x} \right) = \frac{d}{dx} x^{-1} = -1x^{-2} = -\frac{1}{x^2}$$

3.1c Polynomials (Cont.)

Ex $\frac{d}{dx} \sqrt{x} = \frac{d}{dx} x^{\frac{1}{2}} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

This one is worth remembering

$$\boxed{\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}}$$

Algebra

1) $x^{-n} = \frac{1}{x^n}$

2) $x^{\frac{1}{n}} = \sqrt[n]{x}$

3) $x^{\frac{m}{n}} = \sqrt[n]{x^m}$

Ex $\frac{d}{dx} x\sqrt{x} = \frac{d}{dx} x \cdot x^{\frac{1}{2}} = \frac{d}{dx} x^{\frac{3}{2}} = \frac{3}{2} x^{\frac{1}{2}} = \frac{3}{2} \sqrt{x}$

Ex $\frac{d}{dx} x^{\frac{5}{2}} = \frac{5}{2} x^{\frac{3}{2}}$

Ex $\frac{d}{dx} \frac{x^2 - 3x}{\sqrt{x}} = \frac{d}{dx} [x^{2-\frac{1}{2}} - 3x^{1-\frac{1}{2}}]$
 $= \frac{d}{dx} [x^{\frac{3}{2}} - 3x^{\frac{1}{2}}]$
 $= \frac{3}{2} x^{\frac{1}{2}} - \frac{3}{2} x^{-\frac{1}{2}}$
 $= \frac{3}{2} \sqrt{x} - \frac{3}{2\sqrt{x}}$

Ex $\frac{d}{dx} x[x^2-1]^2 = \frac{d}{dx} x(x^4-2x^2+1)$
 $= \frac{d}{dx} (x^5-2x^3+x) = 5x^4-6x^2+1$

3.1d Exponentials

Let $f(x) = a^x$ $a > 0$

$$f'(x) = \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \rightarrow 0} \frac{a^x a^h - a^x}{h}$$

$$= \lim_{h \rightarrow 0} a^x \left[\frac{a^h - 1}{h} \right]$$

$$= a^x \lim_{h \rightarrow 0} \left[\frac{a^h - 1}{h} \right]$$

For now, we can use a graphing calculator to evaluate the limit numerically.

Call this limit $\ln a$

$$a=2 \quad \lim_{h \rightarrow 0} \left[\frac{2^h - 1}{h} \right] = 0.693 = \ln 2$$

$$a=3 \quad \lim_{h \rightarrow 0} \left[\frac{3^h - 1}{h} \right] = 1.0986 = \ln 3$$

$$a=4 \quad \lim_{h \rightarrow 0} \left[\frac{4^h - 1}{h} \right] = 1.3863 = \ln 4$$

For some number between 2 and 3, there is a unique number for which the limit above is equal to 1. This number is called e (for Euler.)

$$e = 2.71828 \quad \ln e = 1$$

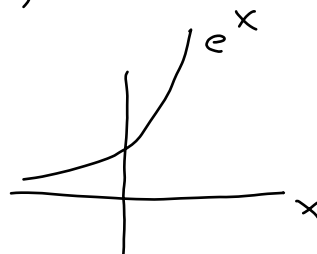
Conclusion:

$$f(x) = a^x$$

$$f'(x) = a^x \ln a$$

$$f(x) = e^x$$

$$f'(x) = e^x$$



The function $f(x) = e^x$ is the only function whose derivative is itself

$$\text{Ex: } \frac{d}{dx} 2^x = 2^x \ln 2$$

$$\frac{d}{dx} 5e^x = 5e^x$$

$$\frac{d}{dx} 7^x = 7^x \ln 7$$

$$\frac{d}{dx} \frac{e^x}{2} = \frac{e^x}{2}$$

Product and Quotient Rules

3.2 Sum and Product Rules

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Differentiation Rules

$y = f(u), u=g(x), v=h(x)$

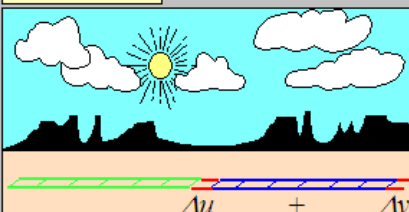
$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$ **Ex1**

$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

$\frac{d}{dx} \frac{u}{v} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Sum Rule



$\Delta(u+v) = \Delta u + \Delta v$

$\frac{\Delta(u+v)}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}$

Taking the limit as Δx goes to 0

$\frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

Visual "proof" of sum rule

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Differentiation Rules

$y = f(u), u=g(x), v=h(x)$

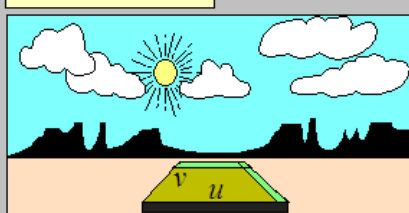
$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$ **Ex2**

$\frac{d}{dx} \frac{u}{v} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Product Rule



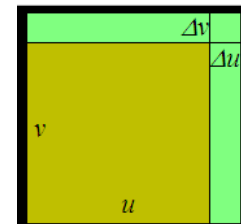
$\Delta(uv) = u\Delta v + v\Delta u + \Delta u\Delta v$

$\frac{\Delta(uv)}{\Delta x} = u \frac{\Delta v}{\Delta x} + v \frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x} \Delta v$

Taking the limit as Δx goes

$\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$

Visual "proof" of Product rule.



3.2a Quotient Rule

Proof of Quotient Rule

Ex 1: $y = x^2 e^x$
 $y' = x^2 \frac{d}{dx} e^x + e^x \frac{d}{dx} (x^2)$
 $= x^2 e^x + e^x 2x$
 $= e^x (x^2 + 2x)$

Ex 2: $y = \frac{e^x + 1}{e^x - 1}$ $y' = \frac{(e^x - 1)(e^x + 1)' - (e^x + 1)'(e^x - 1)}{(e^x - 1)^2}$
 $y' = \frac{(e^x - 1)e^x - (e^x - 1)e^x}{(e^x - 1)^2}$
 $y' = \frac{e^x [(e^x - 1) - (e^x - 1)]}{(e^x - 1)^2} = \frac{-2e^x}{(e^x - 1)^2}$

Ex 3 $y = \frac{x}{x+1}$
 $y' = \frac{(x+1)(x)' - x(x+1)'}{(x+1)^2} = \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)^2}$

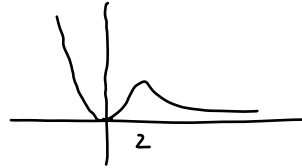
3.2a Product and Quotient Rules (Cont.)

Ex $y = x^2 e^{-x}$ Find y'

$$y = \frac{x^2}{e^x}$$

$$y' = \frac{(e^x)(x^2)' - x^2(e^x)'}{e^{2x}}$$

$$= \frac{e^x(2x) - x^2 e^x}{e^{2x}} = \frac{e^x [2x - x^2]}{e^{2x}} = x(2-x)e^{-x}$$



Note that the function has 0-slope at $x=0$ and $x=2$

Ex $y = \frac{x}{x + \frac{c}{x}}$. Multiply top and bottom by x

$$y = \frac{x^2}{x^2 + c}$$

$$y' = \frac{(x^2+c)(x^2)' - x^2(x^2+c)'}{(x^2+c)^2} = \frac{(x^2+c)(2x) - x^2(2x)}{(x^2+c)^2}$$

$$y' = \frac{2x[(x^2+c) - x^2]}{(x^2+c)^2} = \frac{2xc}{(x^2+c)^2}$$

Ex $y = \frac{x^2 + 4x + 3}{\sqrt{x}}$

Better to avoid the quotient rule

$$y = \frac{x^2 + 4x + 3}{x^{1/2}}$$

$$y' = x^{3/2} + 4x^{1/2} + 3x^{-1/2}$$

$$y' = \frac{3}{2}x^{1/2} + \frac{4}{2}x^{-1/2} - \frac{3}{2}x^{-3/2} = \frac{3\sqrt{x}}{2} + \frac{2}{\sqrt{x}} - \frac{3}{2x\sqrt{x}}$$

$$y' = \frac{3x^2 + 4x - 3}{2x\sqrt{x}}$$

← Common Denom.

Rates of Change in the Sciences

3.3 Rates of Change in the Sciences

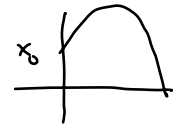
Physics: $x = x(t)$ Position
 $\frac{dx}{dt} = v(t)$ Velocity
 $\frac{dv}{dt} = a(t)$ Acceleration.

Uniform Motion.

A particle is thrown upwards at the surface of the Earth with initial velocity v_0 and initial height x_0 .

Then

$$\begin{aligned} a &= \frac{dv}{dt} = -g & g &= 9.80 \text{ m/s}^2 = 32 \text{ ft/s}^2 \\ v &= -gt + v_0 & \left(\frac{dv}{dt} = -g, \quad v(0) = -g(0) + v_0 = v_0 \right) \\ x &= -\frac{1}{2}gt^2 + v_0t + x_0 & \left(\frac{dx}{dt} = v, \quad x(0) = x_0 \right) \end{aligned}$$



In general, a particle moving with constant acceleration ($a = \text{const}$) satisfies the equation

$$\begin{cases} v = at + v_0 & v(0) = v_0 \\ x = \frac{1}{2}at^2 + v_0t + x_0 & x(0) = x_0 \end{cases}$$

The velocity is a linear function
The path of the particle is parabolic

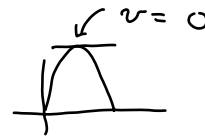
3.3a Rates of Change in the Sciences

Example: A ball is thrown upwards from the ground with a velocity of 80 ft/s. Find

- The maximum height
- The velocity when the height is 96 ft

Sol:

$$\begin{aligned} x_0 &= 0 \\ v_0 &= 80 \end{aligned} \quad \left\{ \begin{aligned} x &= -\frac{1}{2}(32)t^2 + 80t = -16t^2 + 80t \\ v &= -32t + 80 \end{aligned} \right.$$



a) Maximum height: set $v = 0$

$$-32t + 80 = 0$$

$$t = \frac{80}{32} = \frac{10}{4} = \frac{5}{2} = 2.5 \text{ s}$$

b) $x = -16t^2 + 80t = 96 \Rightarrow t^2 - 5t + 6 = 0$

$$16t^2 - 80t + 96 = 0 \Rightarrow (t-3)(t-2) = 0 \quad t = 2, 3$$

$$v(2) = -32(2) + 80 = 16 \text{ UP}$$

$$v(3) = -32(3) + 80 = -16 \text{ Down}$$

Ex: The position of a particle is given by

$$s = t^3 - 4.5t^2 - 7t \quad t \geq 0$$

When does the particle reach a velocity of 5 m/s

Sol $v = 3t^2 - 9t - 7 = 5$

$$3t^2 - 9t - 12 = 0$$

$$t^2 - 3t - 4 = 0$$

$$(t-4)(t+1) = 0 \quad t = 4$$

Ex The charge Q in a wire is given by

$$Q = t^3 - 2t^2 + 6t + 2 \quad (\text{Coulombs})$$

Find the current I when $t = 1$ sec

Sol $I = \frac{dQ}{dt} = 3t^2 - 4t + 6$

$$I(1) = 3 - 4 + 6 = 5 \text{ amps}$$

3.3b Rates of Change in the Sciences

Chemistry

Rates of Chemical Reactions

Example: Consider the reaction



Let $[A]$, $[B]$ and $[C]$ be the concentrations

Suppose that:

$$[A] = [B] = a \text{ moles/L and}$$

$$[C] = \frac{a^2 k t}{a k t + 1} \quad k = \text{const}$$

a) Find the rate of the reaction at time t

Sol

$$\begin{aligned} \frac{d}{dt} [C] &= \frac{(a k t + 1)(a^2 k t)' - a^2 k t [a k t + 1]'}{(a k t + 1)^2} \\ &= \frac{(a k t + 1)(a^2 k) - a^2 k t (a k)}{(a k t + 1)^2} \\ &= \frac{a^2 k [a k t + 1 - a k t]}{(a k t + 1)^2} = \frac{a^2 k}{(a k t + 1)^2} \end{aligned}$$

b) Let $x = [C]$

Show that $\frac{dx}{dt} = k(a-x)^2$

Sol

$$\begin{aligned} k(a-x)^2 &= k \left[a - \frac{a^2 k t}{a k t + 1} \right]^2 = k \left[\frac{a^2 k t + a - a^2 k t}{a k t + 1} \right]^2 \\ &= k \frac{a^2}{(a k t + 1)^2} \\ &= \frac{dx}{dt} \end{aligned}$$

c) What happens as $t \rightarrow \infty$

$$\text{Sol } \lim_{t \rightarrow \infty} [C] = \lim_{t \rightarrow \infty} \frac{a^2 k t}{a k t + 1} = a$$

$$\lim_{t \rightarrow \infty} \frac{d[C]}{dt} = \lim_{t \rightarrow \infty} \frac{a^2 k}{(a k t + 1)^2} = 0$$

As $t \rightarrow \infty$ the number of C molecules approaches the number of A molecules and the reaction rate goes to zero.

Trigonometric Functions

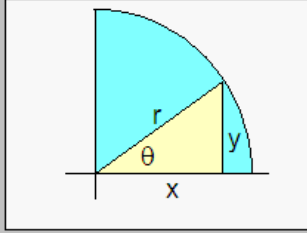
3.4 Trigonometric Functions (Review)

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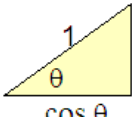
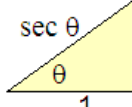
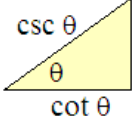
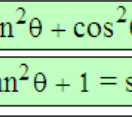
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Trigonometric Functions (1)

Brief Review



$\sin \theta = y/r$	$\csc \theta = r/y$
$\cos \theta = x/r$	$\sec \theta = r/x$
$\tan \theta = y/x$	$\cot \theta = x/y$
$x = r \cos \theta$	$y = r \sin \theta$

$\sec \theta = \frac{1}{\cos \theta}$		$\sin \theta$
$\tan \theta = \frac{\sin \theta}{\cos \theta}$		$\tan \theta$
$\csc \theta = \frac{1}{\sin \theta}$		1
$\cot \theta = \frac{\cos \theta}{\sin \theta}$		1

$\sin 0 = 0$	$\sin^2 \theta + \cos^2 \theta = 1$
$\cos 0 = 1$	$\tan^2 \theta + 1 = \sec^2 \theta$
$\sin \pi/2 = 1$	$\cot^2 \theta + 1 = \csc^2 \theta$
$\cos \pi/2 = 0$	

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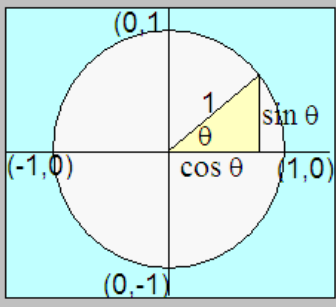
Trigonometric Functions (2)

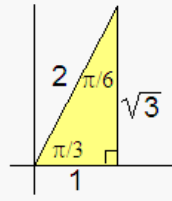
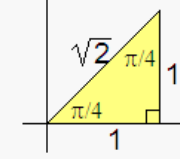
Sum formulas

$\sin(\theta+\phi) = \sin \theta \cos \phi + \sin \phi \cos \theta$

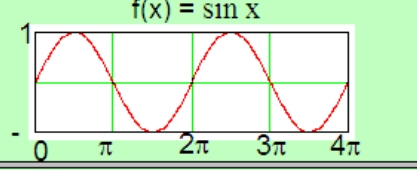
$\cos(\theta+\phi) = \cos \theta \cos \phi - \sin \theta \sin \phi$

Special Angles

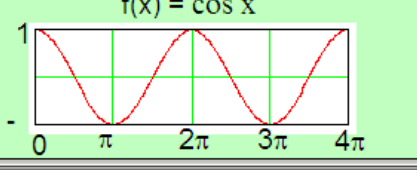


	
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$f(x) = \sin x$



$f(x) = \cos x$



Cont

3.4a Trigonometric Functions

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Derivatives of Trig Functions **Derivative of f(x)=sin x**

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h}$$
$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h}$$
$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{\sin x(\cos h - 1) + \sin h \cos x}{h}$$
$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \sin x \frac{(\cos h - 1)}{h} + \lim_{h \rightarrow 0} \cos x \frac{\sin h}{h}$$
$$\frac{dy}{dx} = \sin x \lim_{h \rightarrow 0} \frac{(\cos h - 1)}{h} + \cos x \lim_{h \rightarrow 0} \frac{\sin h}{h}$$
$$\frac{dy}{dx} = \sin x \cdot 0 + \cos x \cdot 1$$

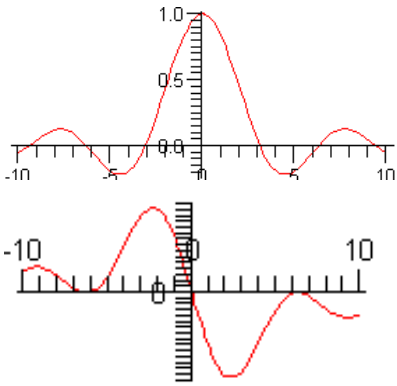
$\frac{d}{dx} \sin x = \cos x$

3.4b Trigonometric Functions

We seek a formula for the derivative of $f(x) = \sin x$

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [-\sin x \cos h + \sin h \cos x - \sin x] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [\sin x (\cos h - 1) + \sin h \cos x] \\
 &= (\sin x) \lim_{h \rightarrow 0} \left(\frac{\cos h - 1}{h} \right) + (\cos x) \lim_{h \rightarrow 0} \frac{\sin h}{h}
 \end{aligned}$$

Use:
 $\sin(x+y) = \sin x \cos y + \sin y \cos x$



Using Maple

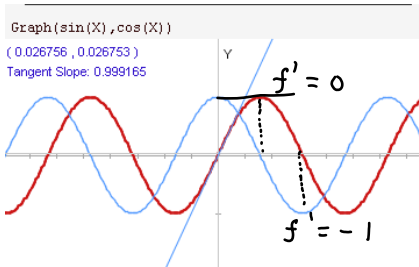
$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{\sin h}{h} &= 1 \\
 \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} &= 0
 \end{aligned}$$

Important limits to remember.

$$\therefore \frac{d}{dx} \sin x = \cos x$$

$$\frac{d}{dx} \cos x = -\sin x$$

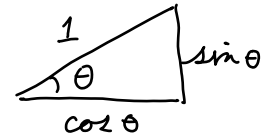
In a similar fashion one can show



The formula for the derivative of $\sin x$ makes sense graphically. Notice that at $x=0$ $\sin x$ has slope = 1

3.4c Trigonometric Functions (Cont.)

$$\begin{aligned}\frac{d}{dx} \tan x &= \frac{d}{dx} \frac{\sin x}{\cos x} = \frac{\cos x (\sin x)' - \sin x (\cos x)'}{\cos^2 x} \\ &= \frac{\cos x \cos x - \sin x (-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$



$$\begin{aligned}\frac{d}{dx} \sec x &= \frac{d}{dx} \frac{1}{\cos x} = \frac{\cos x \cdot (0) - 1(-\sin x)}{\cos^2 x} \\ &= \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} \\ &= \tan x \sec x\end{aligned}$$

Similarly: $\frac{d}{dx} \cot x = -\csc^2 x$

$$\frac{d}{dx} \csc x = -\cot x \csc x$$

Summary

- 1) $\frac{d}{dx} \sin x = \cos x$
- 2) $\frac{d}{dx} \cos x = -\sin x$
- 3) $\frac{d}{dx} \tan x = \sec^2 x$
- 4) $\frac{d}{dx} \csc x = -\csc x \cot x$
- 5) $\frac{d}{dx} \sec x = \sec x \tan x$
- 6) $\frac{d}{dx} \cot x = -\csc^2 x$

Derivatives of Trig Functions	
$\frac{d}{dx} \sin x = \cos x$	
$\frac{d}{dx} \cos x = -\sin x$	
$\frac{d}{dx} \sec x = \sec x \tan x$	
$\frac{d}{dx} \tan x = \sec^2 x$	
$\frac{d}{dx} \cot x = -\csc^2 x$	
$\frac{d}{dx} \csc x = -\csc x \cot x$	

3.4c Trigonometric Functions (Cont.)

Examples: Differentiate

1) $f(x) = x \sin x$

$$f'(x) = x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x)$$

$$= x \cos x + \sin x$$

2) $f(x) = e^x \sin x$

$$f'(x) = e^x (\sin x)' + \sin x (e^x)'$$

$$= e^x \cos x + \sin x \cdot e^x$$

$$= e^x (\cos x + \sin x)$$

3) $y = \frac{\tan x}{x}$ $y' = \frac{x(\tan x)' - \tan x(x)'}{x^2}$

$$= \frac{1}{x^2} [x \sec^2 x - \tan x]$$

4) $y = \csc x \cot x$ $y' = \csc x (\cot x)' + \cot x (\csc x)'$

$$= \csc x (-\csc^2 x) + \cot x (-\csc x \cot x)$$

$$= -\csc x [\csc^2 x + \cot^2 x]$$

Evaluate the given limits

5) $\lim_{x \rightarrow 0} \frac{\sin 5x}{x} = \lim_{5x \rightarrow 0} 5 \cdot \frac{\sin 5x}{5x}$ Let $h = 5x$

$$= 5 \lim_{h \rightarrow 0} \frac{\sin h}{h} = 5$$

38) 6) $\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\sin \theta} = \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} \cdot \lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta} = 0 \cdot \frac{1}{1} = 0$

7) $\lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 5x} = \lim_{x \rightarrow 0} \frac{5 \sin 3x}{3 \sin 5x} \cdot \frac{5x}{5x} = \frac{5}{3} \cdot \frac{1}{1} \cdot \frac{1}{5} = \frac{1}{3}$

8) $\lim_{x \rightarrow 0} \frac{\tan x}{4x} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{1}{4 \cos x} = 1 \cdot \frac{1}{4} = \frac{1}{4}$

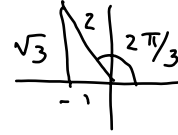
3.4d Trigonometric Functions (Cont.) - Questions

31) Ex: Let $x(t) = 8 \sin t$ Find the velocity
at $t = 2\pi/3$

Sol:

$$v(t) = x'(t) = 8 \cos t$$

$$v\left(\frac{2\pi}{3}\right) = 8 \cos\left(\frac{2\pi}{3}\right) = 8\left(-\frac{1}{2}\right) = -4$$



11) Ex: $f(x) = \frac{\sec \theta}{1 + \sec \theta}$

$$f'(x) = \frac{(1 + \sec \theta)(\sec \theta \tan \theta) - \sec \theta (\sec \theta \tan \theta)}{(1 + \sec \theta)^2}$$

$$= \frac{\sec \theta \tan \theta [1 + \sec \theta - \sec \theta]}{(1 + \sec \theta)^2}$$

$$= \frac{\sec \theta \tan \theta}{(1 + \sec \theta)^2}$$

Chain Rule

3.5 Chain Rule

Theorem: If $y = f(u)$ and $u = g(x)$, then

or
$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x) \quad \text{- Newton's notation}$$

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} \quad \text{- Leibnitz notation}$$

This is called the Chain rule factor

The chain rule is very powerful. It can be used to generalize the formulas for the derivatives of the elementary functions

Ex If $y = u^n$ then $\frac{dy}{dx} = nu^{n-1} \frac{du}{dx}$
 $u = g(x)$

If $y = \sin u$ then $\frac{dy}{dx} = \cos u \frac{du}{dx}$ etc.
 $u = g(x)$

Ex) $\frac{d}{dx} (x^3+2)^4 = 4(x^3+2)^3 \frac{d}{dx} (x^3+2) = 12x^2 (x^3+2)^3$

2) $\frac{d}{dx} \sin(x^4) = \cos x^4 \cdot \frac{d}{dx} (x^4) = 4x^3 \cos x^4$

3) $\frac{d}{dx} e^{3x+2} = e^{3x+2} \frac{d}{dx} (3x+2) = 3e^{3x+2}$

4) $\frac{d}{dx} \cos(e^x) = -\sin(e^x) \frac{d}{dx} (e^x) = -e^x \sin(e^x)$

5) $\frac{d}{dx} \sec(5x) = \sec(5x) \tan(5x) \frac{d}{dx} (5x)$
 $= 5 \sec 5x \tan 5x$

6) $\frac{d}{dx} \sqrt{x^2+1} = \frac{1}{2} (x^2+1)^{-\frac{1}{2}} \frac{d}{dx} (x^2+1) = \frac{x}{\sqrt{x^2+1}}$

7) $\frac{d}{dx} \tan(\sin x) = \sec^2(\sin x) \cdot \frac{d}{dx} (\sin x)$
 $= \cos x \cdot \sec^2(\sin x)$

8) $\frac{d}{dx} x e^{-x^2} = e^{-x^2} \frac{d}{dx} (x) + x e^{-x^2} \frac{d}{dx} (-x^2) = e^{-x^2} [1 - 2x^2]$

3.5a Chain Rule

More Examples:

$$\begin{aligned} 1) \frac{d}{dx} \sin^3(x^2) &= 3 \sin^2(x^2) \frac{d}{dx} (\sin x^2) \\ &= 3 \sin^2(x^2) \cdot \cos(x^2) \cdot \frac{d}{dx} (x^2) \\ &= 6x \sin^2(x^2) \cos(x^2) \end{aligned}$$

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$$\begin{aligned} 2) \frac{d}{dx} e^{x \cos x} &= e^{x \cos x} \cdot \frac{d}{dx} (x \cos x) \\ &= e^{x \cos x} [\cos x - x \sin x] \end{aligned}$$

$$\begin{aligned} 3) y &= (2x+1)^3 (3x-2)^5 \\ y' &= (2x+1)^3 \frac{d}{dx} (3x-2)^5 + (3x-2)^5 \frac{d}{dx} (2x+1)^3 \\ &= (2x+1)^3 \cdot 5(3x-2)^4 \cdot 3 + (3x-2)^5 \cdot 3(2x+1)^2 \cdot 2 \\ &= (2x+1)^2 (3x-2)^4 [(2x+1)(15) + (3x-2)6] \\ &= (2x+1)^2 (3x-2)^4 [30x+15+9x-12] \\ &= (2x+1)^2 (3x-2)^4 (39x-3) \\ &= 3(2x+1)^2 (3x-2)^4 (13x-1) \end{aligned}$$

$$\begin{aligned} 4) y &= \sqrt{x+\sqrt{x}} \\ y' &= \frac{1}{2\sqrt{x+\sqrt{x}}} \cdot \frac{d}{dx} (x+\sqrt{x}) = \frac{1}{2\sqrt{x+\sqrt{x}}} \left(1 + \frac{1}{2\sqrt{x}}\right) \end{aligned}$$

$$\begin{aligned} 5) y &= \sin^2(\cos 5x) \\ y' &= 2 \sin(\cos 5x) \cdot \frac{d}{dx} (\cos 5x) \\ &= 2 \sin(\cos 5x) (-\sin 5x) \frac{d}{dx} (5x) \\ &= -10 \sin(\cos 5x) \sin 5x. \end{aligned}$$

3.5b Chain Rule

Problems

25) 1. $y = \sqrt{\frac{z-1}{z+1}}$ Note $\frac{d}{dx} \sqrt{u} = \frac{1}{2\sqrt{u}} \frac{du}{dx}$

$$y' = \frac{1}{2\sqrt{\frac{z-1}{z+1}}} \frac{d}{dz} \frac{z-1}{z+1}$$

$$= \frac{1}{2\sqrt{\frac{z-1}{z+1}}} \frac{(z+1)(1) - (z-1)(1)}{(z+1)^2} = \frac{1}{2\sqrt{\frac{z-1}{z+1}}} \frac{2}{(z+1)^2}$$

$$= \frac{1}{(z+1)^2} \sqrt{\frac{z+1}{z-1}}$$

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2. $y = \frac{r}{\sqrt{r^2+1}}$

$$y' = \frac{\sqrt{r^2+1} (r)' - r (\sqrt{r^2+1})'}{r^2+1}$$

$$= \frac{\sqrt{r^2+1} - r \left(\frac{1}{2\sqrt{r^2+1}} \right) \cdot 2r}{r^2+1} =$$

$$= \left[\frac{\sqrt{r^2+1} - \frac{r^2}{\sqrt{r^2+1}}}{r^2+1} \right] \frac{\sqrt{r^2+1}}{\sqrt{r^2+1}} = \frac{r^2+1 - r^2}{(r^2+1)^{3/2}}$$

$$= \frac{1}{(r^2+1)^{3/2}}$$

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3. $y = \sec^2 x + \tan^2 x$

$$y' = 2 \sec x (\sec x)' + 2 \tan x (\tan x)'$$

$$= 2 \sec x (\sec x \tan x) + 2 \tan x (\sec^2 x)$$

$$= 2 \sec^2 x \{ \tan x + \tan x \}$$

$$= 4 \sec^2 x \tan x$$

Implicit Differentiation

3.6 Implicit Differentiation

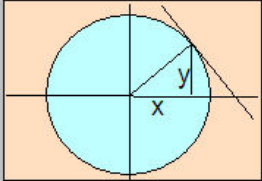
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2.4 Implicit Differentiation

Example $x^2 + y^2 = 1$

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}1$$

$$2x + 2y\frac{dy}{dx} = 0 \quad | \quad 2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{2y} \quad \boxed{\frac{dy}{dx} = -\frac{x}{y}}$$


$y = \pm\sqrt{1-x^2}$
 $y = \pm(1-x^2)^{1/2}$

$$\frac{dy}{dx} = \pm\frac{1}{2}(1-x^2)^{-1/2} \frac{d}{dx}(1-x^2)$$

$$\frac{dy}{dx} = \pm\frac{1}{2} \frac{1}{\sqrt{1-x^2}} (-2x)$$

$$\frac{dy}{dx} = \frac{-x}{\pm\sqrt{1-x^2}} \quad \boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

Graph

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2.6 Implicit Differentiation (2) **Example** $x^4 + 4x^2y^2 + y^4 = 0$

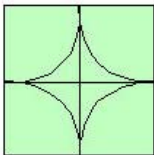
Example

$$x^{2/3} + y^{2/3} = a^{2/3}$$

$$\frac{d}{dx}x^{2/3} + \frac{d}{dx}y^{2/3} = \frac{d}{dx}a^{2/3}$$

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$x^{-1/3} + y^{-1/3}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}}$$


$\frac{d}{dx}x^4 + 4\frac{d}{dx}(x^2y^2) + \frac{d}{dx}y^4 = 0$

$$4x^3 + 4\left(2xy^2 + x^2 \cdot 2y\frac{dy}{dx}\right) + 4y^3\frac{dy}{dx} = 0$$

$$4x^3 + 8xy^2 + 8x^2y\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 0$$

$$(8x^2y + 4y^3)\frac{dy}{dx} = -(4x^3 + 8xy^2)$$

$$\frac{dy}{dx} = -\frac{(4x^3 + 8xy^2)}{8x^2y + 4y^3} \quad \boxed{\frac{dy}{dx} = -\frac{x^3 + 2xy^2}{2x^2y + y^3}}$$

3.6a Implicit Differentiation

Example

9) 1) $xy = \cot(xy)$
 $\frac{d}{dx}(xy) = \frac{d}{dx} \cot(xy) \Rightarrow y + x \frac{dy}{dx} = -\csc^2(xy) \left[y + x \frac{dy}{dx} \right]$
 $y + xy' = -y \csc^2(xy) - xy' \csc^2(xy)$
 $[x + x \csc^2(xy)] y' = -y - y \csc^2(xy)$
 $y' = \frac{-y(1 + \csc^2(xy))}{x(1 + \csc^2(xy))} = \frac{-y}{x}$

2) Find the equation of the tangent line to $y^2 = x^3(2-x)$ at the point $(1,1)$

Sol

$$y^2 = 2x^3 - x^4$$

$$2y \frac{dy}{dx} = 6x^2 - 4x^3 = 2x^3(3x - 4)$$

$$\frac{dy}{dx} = \frac{x^3(3x-4)}{y} \quad \left. \frac{dy}{dx} \right|_{(1,1)} = \frac{1(3-4)}{1} = -1 = m$$

$$y - 1 = -1(x - 1)$$

$$y = -x + 1 + 1 \quad y = -x + 2$$

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3. $4 \cos x \sin y = 1$
 $4 \cos x \frac{d}{dx} \sin y + 4 \frac{d}{dx} (\cos x) \sin y = 0$
 $4 \cos x \cos y \frac{dy}{dx} - 4 \sin x \sin y = 0$

$$\cos x \cos y \frac{dy}{dx} = \sin x \sin y$$

$$\frac{dy}{dx} = \frac{\sin x \sin y}{\cos x \cos y} = \tan x \tan y$$

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4. $e^{x^2 y} = x + y$
 $e^{x^2 y} (2xy + x^2 \frac{dy}{dx}) = 1 + \frac{dy}{dx}$
 $[x^2 e^{x^2 y} - 1] \frac{dy}{dx} = 1 - 2xy e^{x^2 y} \Rightarrow y' = \frac{1 - 2xy e^{x^2 y}}{x^2 e^{x^2 y} - 1}$

3.6b Implicit Differentiation (Cont.)

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$$\begin{aligned}
 5) \quad \sqrt{xy} &= 1 + x^2y \\
 \frac{d}{dx} [xy' + x] &= x^2y' + 2xy \\
 xy' + x &= 2x^2\sqrt{xy}y' + 2xy \cdot 2\sqrt{xy} \\
 (x - 2x^2\sqrt{xy})y' &= 4xy\sqrt{xy} - x \\
 y' &= \frac{4xy\sqrt{xy} - x}{x - 2x^2\sqrt{xy}}
 \end{aligned}$$

3.6c Implicit Differentiation (Cont.)

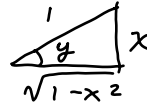
Inverse Trigonometric Functions

1) Let $y = \sin^{-1} x$, find y'

Sol. $\sin y = x$

$$\frac{d}{dx} \sin y = \frac{d}{dx} (x) \Rightarrow (\cos y) \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

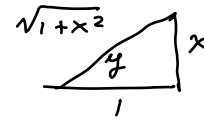


2) Let $y = \tan^{-1} x$, find y'

Sol. $\tan y = x$

$$\frac{d}{dx} \tan y = \frac{d}{dx} (x) \Rightarrow \sec^2 y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+x^2}$$



Basic Formulas:

$$\frac{d}{dx} \sin^{-1} u = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$$

$$\frac{d}{dx} \tan^{-1} u = \frac{1}{1+u^2} \frac{du}{dx}$$

$$\frac{d}{dx} \sec^{-1} u = \frac{1}{u\sqrt{u^2-1}} \frac{du}{dx}$$

Examples

1) $y = \sin^{-1}(x^2)$ $y' = \frac{1}{\sqrt{1-x^4}} \frac{d}{dx} x^2 = \frac{2x}{\sqrt{1-x^4}}$

2) $y = \tan^{-1} 5x$ $y' = \frac{1}{1+25x^2} \frac{d}{dx} (5x) = \frac{5}{1+25x^2}$

3) $y = \sin^{-1}(e^x)$ $y' = \frac{1}{\sqrt{1+e^{2x}}} \frac{d}{dx} e^x = \frac{e^x}{\sqrt{1+e^{2x}}}$

4) $y = \tan^{-1}(\cos \theta)$ $y' = \frac{1}{1+\cos^2 \theta} \frac{d}{d\theta} \cos \theta = \frac{-\sin \theta}{1+\cos^2 \theta}$

3.6c Implicit Differentiation (Cont.)

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$$5. \quad y = x \cos^{-1} x - \sqrt{1-x^2}$$

$$y' = \cos^{-1} x + x \frac{1}{-\sqrt{1-x^2}} - \frac{1}{2\sqrt{1-x^2}} (-2x) = \cos^{-1} x$$

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$$6. \quad y = \tan^{-1} [x - \sqrt{1+x^2}]$$

$$y' = \frac{1}{1+(x-\sqrt{1+x^2})^2} \cdot \frac{d}{dx} (x - \sqrt{1+x^2})$$

$$= \frac{1}{1+(x-\sqrt{1+x^2})^2} \left[1 - \frac{1}{2\sqrt{1+x^2}} \cdot 2x \right]$$

$$= \frac{1}{1+[x-\sqrt{1+x^2}]^2} \left[1 - \frac{x}{\sqrt{1+x^2}} \right]$$

*Not much one
can do to
simplify*

Higher Order Derivatives

3.7 Higher Order Derivatives

Notation

Let $y = f(x)$

First derivative: $y' = f'(x) = \frac{dy}{dx} = Df(x) = Dy$

Second derivative $y'' = f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D^2f(x)$

If $x = f(t)$ represents position
 $\frac{dx}{dt} = v(t)$ represents velocity

$\frac{d^2x}{dt^2} = a(t)$ represents acceleration.

$\frac{d^3x}{dt^3} = j(t)$ represents a "jerk".

Examples

1) $y = x^3 - 5x^2 + 2x + 7$
 $y' = 3x^2 - 10x + 2$
 $y'' = 6x - 10$
 $y''' = 6$
 $y^{(4)} = 0$

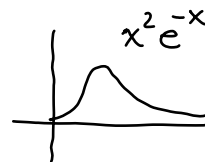
2. $y = x^4$
 $y' = 4x^3$
 $y'' = (4)(3)x^2$
 $y''' = (4)(3)(2)x$
 $y^{(4)} = (4)(3)(2)(1) = 4!$

3) $f(x) = x^n$
 $f^{(n)}(x) = n!$

4) $f(x) = e^x$
 $f^{(n)}(x) = e^x$

5) $f(x) = \sin x$ $f^{(4)}(x) = \sin x$ "cycle" of period 4
 $f'(x) = \cos x$ $f^{(5)}(x) = \cos x$
 $f''(x) = -\sin x$ \vdots
 $f'''(x) = -\cos x$

6) $y = x^2 e^{-x}$
 $y' = x^2 e^{-x}(-1) + 2x e^{-x} = e^{-x}(-x^2 + 2x)$
 $y'' = e^{-x}(-2x + 2) + (-x^2 + 2x)e^{-x}(-1)$
 $= e^{-x}[-2x + 2 + x^2 - 2x]$
 $= e^{-x}[x^2 - 4x + 2]$



3.7 Higher Order Derivatives (Cont.)

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7. The position of a particle is given by $s(t) = t^3 - 12t^2 + 3t$, $t \geq 0$. Find

a) The acceleration after 3 seconds

$$s'(t) = v(t) = 3t^2 - 24t + 3$$

$$s''(t) = a(t) = 6t - 24 \quad a(3) = 6(3) - 24 = -6$$

b) When is the particle speeding up?

$$a(t) = 6t - 24 > 0$$

$$6t > 24$$

$$t > 4$$

Speeding when $t > 4$

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Harmonic Motion

8. The position of a mass m attached to a spring with "stiffness" k is given by

$$y(t) = A \sin \omega t \quad (A, \omega = \sqrt{\frac{k}{m}} \text{ const.})$$

a) Find the velocity and acceleration

$$v(t) = y'(t) = A\omega \cos \omega t$$

$$a(t) = y''(t) = -A\omega^2 \sin \omega t$$

b) Show that the acceleration is proportional to the displacement

$$a = -k\omega^2 \sin \omega t$$

$$= -\omega^2 (A \sin \omega t)$$

$$a = -\omega^2 y$$

c) Show that the force is given by $F = -ky$

$$F = ma = -m\omega^2 y$$

$$= -m\left(\sqrt{\frac{k}{m}}\right)^2 y = -m\frac{k}{m} y \Rightarrow \boxed{F = -ky}$$

Logarithmic Functions

3.8 Logarithmic Functions

Recall the definitions

$$1) y = \ln x \Leftrightarrow e^y = x \quad | \quad 2) y = \log_a x \Leftrightarrow a^y = x$$

By implicit differentiation we get

$$\frac{d}{dx} e^y = \frac{d}{dx} (x)$$

$$e^y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

$$\therefore \frac{d}{dx} \ln x = \frac{1}{x}$$

$$\Rightarrow \boxed{\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}}$$

$$\ln a^y = \ln x$$

$$y \ln a = \ln x$$

$$(\ln a) \frac{dy}{dx} = \frac{d}{dx} \ln x$$

$$\frac{dy}{dx} = \frac{1}{\ln a} \frac{1}{x}$$

$$\Rightarrow \boxed{\frac{d}{dx} \log_a u = \frac{1}{u \ln a} \frac{du}{dx}}$$

I prefer to rewrite $\log_a x$ in terms of natural logarithms rather than memorizing one more formula

Examples

1. $y = \ln(\sec x)$

$$y' = \frac{1}{\sec x} \frac{d}{dx} (\sec x)$$

$$= \frac{1}{\sec x} \cdot \sec x \tan x$$

$$= \tan x$$

$$\Rightarrow \boxed{\frac{d}{dx} \ln(\sec x) = \tan x}$$

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2. $y = \ln(\sec x + \tan x)$

$$y' = \frac{1}{\sec x + \tan x} \cdot \frac{d}{dx} (\sec x + \tan x)$$

$$= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$

$$= \frac{\sec x (\tan x + \sec x)}{\sec x + \tan x} = \boxed{\sec x}$$

3.8a Logarithmic Functions (Cont.)

The algebraic properties of $\ln x$ are often useful when computing derivatives

- a) $\ln(uv) = \ln u + \ln v$
- b) $\ln(u/v) = \ln u - \ln v$
- c) $\ln u^v = v \ln u$
- d) $\ln e^u = u$

Examples

$$1. \quad y = \ln \sqrt{x^3 + 9} = \ln (x^3 + 9)^{1/2} = \frac{1}{2} \ln (x^3 + 9)$$

$$y' = \frac{1}{2} \cdot \frac{1}{x^3 + 9} \cdot 3x^2 = \frac{3x^2}{2(x^3 + 9)}$$

$$2. \quad y = \ln \sqrt{\frac{x-1}{x+1}} = \frac{1}{2} \ln \left(\frac{x-1}{x+1} \right) = \frac{1}{2} [\ln(x-1) - \ln(x+1)]$$

$$y' = \frac{1}{2} \left[\frac{1}{x-1} - \frac{1}{x+1} \right] = \frac{1}{2} \left[\frac{x+1 - (x-1)}{x^2 - 1} \right] = \frac{1}{x^2 - 1}$$

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$$3. \quad y = \ln [e^{-x} + xe^{-x}] = \ln [e^{-x}(1+x)]$$

$$= \ln e^{-x} + \ln(1+x) = -x + \ln(1+x)$$

$$y' = -1 + \frac{1}{1+x} = \frac{-(1+x) + 1}{1+x} = \frac{-x}{1+x}$$

12

$$4. \quad y = \ln [x + \sqrt{x^2 - 1}]$$

$$y' = \frac{1}{x + \sqrt{x^2 - 1}} \cdot \frac{d}{dx} (x + \sqrt{x^2 - 1}) = \frac{1}{x + \sqrt{x^2 - 1}} \cdot \left(1 + \frac{x}{\sqrt{x^2 - 1}} \right)$$

$$= \frac{1}{x + \sqrt{x^2 - 1}} \cdot \frac{\sqrt{x^2 - 1} + x}{\sqrt{x^2 - 1}} = \frac{1}{\sqrt{x^2 - 1}} = \cosh^{-1} x \quad (3.9)$$

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$$5. \quad f(x) = \ln \frac{(2x+3)^3}{(3x-1)^4} = \ln (2x+3)^3 - \ln (3x-1)^4$$

$$= 3 \ln (2x+3) - 4 \ln (3x-1)$$

$$f'(x) = \frac{3}{2x+3} \cdot \frac{d}{dx} (2x+3) - \frac{4}{3x-1} \cdot \frac{d}{dx} (3x-1)$$

$$= \frac{6}{2x+3} - \frac{12}{3x-1}$$

7

$$6. \quad y = \sqrt[5]{\ln x} \quad y' = \frac{1}{5} (\ln x)^{-4/5} \cdot \frac{1}{x} = \frac{1}{5x \sqrt[5]{\ln^4 x}}$$

3.8b Logarithmic Functions (Cont.)

$$7. y = \ln \frac{e^x \sqrt{x+2}}{(x-3)^2 (2x+1)^5}$$

$$\begin{aligned} y &= \ln [e^x (x+2)^{1/2}] - \ln [(x-3)^2 (2x+1)^5] \\ &= \ln e^x + \ln (x+2)^{1/2} - \ln (x-3)^2 - \ln (2x+1)^5 \\ &= x + \frac{1}{2} \ln (x+2) - 2 \ln (x-3) - 5 \ln (2x+1) \end{aligned}$$

$$y' = 1 + \frac{1}{2(x+2)} - \frac{2}{x-3} - \frac{10}{2x+1}$$

5

$$8. y = \frac{\ln u}{1 + \ln 2u}$$

$$\begin{aligned} y' &= \frac{(1 + \ln 2u)(\ln u)' - \ln u (1 + \ln 2u)'}{(1 + \ln 2u)^2} \\ &= \frac{(1 + \ln 2u)(1/u) - \ln u (1/2u) \cdot 2}{(1 + \ln 2u)^2} \cdot \frac{u}{u} \\ &= \frac{(1 + \ln 2u) - \ln u}{(1 + \ln 2u)^2} = \frac{1 + \ln 2 + \ln u - \ln u}{(1 + \ln 2u)^2} \\ &= \frac{1 + \ln 2}{(1 + \ln 2u)^2} \end{aligned}$$

$$9. y = \log_2 |\sin x| \Rightarrow 2^y = \sin x \Rightarrow y \ln 2 = \ln(\sin x)$$

$$y' \ln 2 = \frac{1}{\sin x} \cdot \frac{d}{dx} (\sin x) = \frac{\cos x}{\sin x} = \cot x$$

$$y' = \frac{\cot x}{\ln 2}$$

$$10. y = 3^{4x+2} \quad \ln y = (4x+2) \ln 3$$

$$\frac{1}{y} y' = 4 \ln 3$$

$$y' = (4 \ln 3) 3^{4x+2}$$

3.8c Logarithmic Differentiation

This process is useful when we have situations of the following types

- 1) $y = a^{f(x)}$
- 2) $y = [f(x)]^{g(x)}$
- 3) $y =$ Expression with more than two products, quotients or radicals

Examples

$$1. \quad y = 2^{x^3}$$

$$\ln y = \ln 2^{x^3} = x^3 \ln 2$$

$$\frac{1}{y} \frac{dy}{dx} = 3x^2 \ln 2$$

Note

$$\frac{d}{dx} \ln y = \frac{1}{y} \frac{dy}{dx}$$

$$y' = (3x^2 \ln 2) y = (3x^2 \ln 2) 2^{x^3}$$

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$$2. \quad y = \sqrt[4]{\frac{x^2+1}{x^2-1}} \quad \ln y = \ln \left[\frac{x^2+1}{x^2-1} \right]^{1/4} = \frac{1}{4} \ln \left[\frac{x^2+1}{x^2-1} \right]$$

$$\ln y = \frac{1}{4} [\ln(x^2+1) - \ln(x^2-1)]$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{4} \left[\frac{2x}{x^2+1} - \frac{2x}{x^2-1} \right] = \frac{2x}{4} \left[\frac{x^2-1 - (x^2+1)}{x^4-1} \right] = \frac{-x}{x^4-1}$$

$$\frac{dy}{dx} = \frac{-x}{x^4-1} \cdot y = \frac{-x}{x^4-1} \sqrt[4]{\frac{x^2+1}{x^2-1}}$$

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$$3. \quad y = (\sin x)^x \quad \ln y = x \ln(\sin x)$$

$$\frac{1}{y} \frac{dy}{dx} = \ln(\sin x) + x \cdot \frac{1}{\sin x} \cos x = \ln(\sin x) + x \cot x$$

$$\frac{dy}{dx} = [\ln(\sin x) + x \cot x] (\sin x)^x$$

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$$4. \quad y = (\ln x)^x \quad \ln y = x \ln(\ln x)$$

$$\frac{1}{y} \frac{dy}{dx} = \ln(\ln x) + x \cdot \frac{1}{\ln x} \cdot \frac{d}{dx} \ln x = \ln(\ln x) + \frac{1}{\ln x}$$

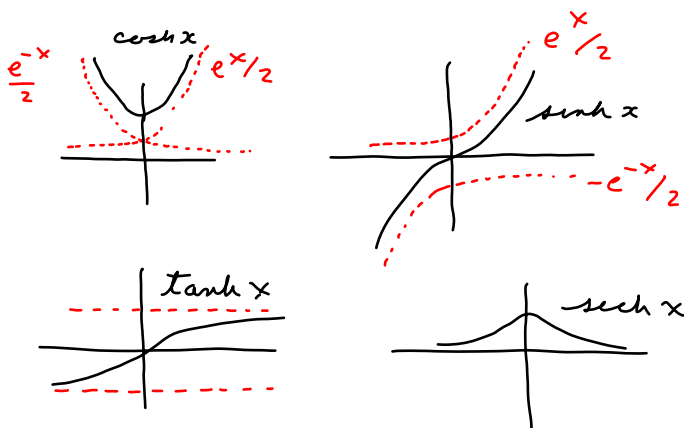
$$\frac{dy}{dx} = \left[\ln(\ln x) + \frac{1}{\ln x} \right] (\ln x)^x$$

Hyperbolic Functions

3.9 Hyperbolic Functions

Definition

1. $\cosh x = \frac{e^x + e^{-x}}{2}$
2. $\sinh x = \frac{e^x - e^{-x}}{2}$
- 3) $\tanh x = \frac{\sinh x}{\cosh x}$
- 4) $\operatorname{sech} x = \frac{1}{\cosh x}$



Basic Identity

$$\begin{aligned} \cosh^2 x - \sinh^2 x &= \left[\frac{e^x + e^{-x}}{2} \right]^2 - \left[\frac{e^x - e^{-x}}{2} \right]^2 \\ &= \frac{1}{4} [e^{2x} + 2e^x e^{-x} + e^{-2x} - (e^{2x} - 2e^x e^{-x} + e^{-2x})] \\ &= \frac{1}{4} [4e^x e^{-x}] = \frac{1}{4} \cdot 4 = 1 \end{aligned}$$

$$\cosh^2 x - \sinh^2 x = 1$$

Applications:

1. Hanging cable
 $y = c + a \cosh(x/a)$



2. Solitary waves (Solitons)

$$y = \operatorname{sech}^2 x$$



Single waves on long, narrow channels

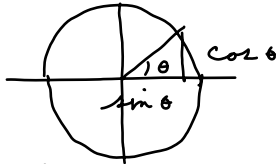
3.9

$$\begin{aligned} \text{Ex: } \sinh(\ln 2) &= \frac{1}{2} [e^{\ln 2} - e^{-\ln 2}] \\ &= \frac{1}{2} \left[2 - \frac{1}{2} \right] = \frac{1}{2} \left(\frac{3}{2} \right) = \frac{3}{4} \end{aligned}$$

3.9b Hyperbolic Functions (Cont.)

Geometry.

Circular Functions



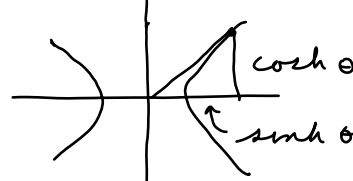
$$x^2 + y^2 = 1$$

$$x = \cos \theta \quad y = \sin \theta$$

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

Hyperbolic Functions



$$x^2 - y^2 = 1$$

$$x = \cosh \theta \quad y = \sinh \theta$$

$$\cosh^2 \theta - \sinh^2 \theta = 1$$

$$1 - \tanh^2 \theta = \operatorname{sech}^2 \theta$$

Derivatives:

$$1. \quad \frac{d}{dx} \cosh x = \frac{d}{dx} \frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$2. \quad \frac{d}{dx} \sinh x = \frac{d}{dx} \frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\begin{aligned} 3) \quad \frac{d}{dx} \tanh x &= \frac{d}{dx} \frac{\sinh x}{\cosh x} = \\ &= \frac{\cosh x (\sinh x)' - \sinh x (\cosh x)'}{\cosh^2 x} \\ &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x \end{aligned}$$

$$\begin{aligned} 4) \quad \frac{d}{dx} \operatorname{sech} x &= \frac{d}{dx} [\cosh x]^{-1} = -(\cosh x)^{-2} \sinh x \\ &= -\frac{\sinh x}{\cosh^2 x} = -\frac{1}{\cosh x} \frac{\sinh x}{\cosh x} \\ &= -\operatorname{sech} x \tanh x \end{aligned}$$

3.9c Hyperbolic Functions (Cont.)

Summary:

1. $\frac{d}{dx} \cosh u = \sinh u \frac{du}{dx}$
2. $\frac{d}{dx} \sinh u = \cosh u \frac{du}{dx}$
3. $\frac{d}{dx} \tanh u = \operatorname{sech}^2 u \frac{du}{dx}$
4. $\frac{d}{dx} \operatorname{sech} u = -\operatorname{sech} u \tanh u \frac{du}{dx}$

Examples

1. $y = \tanh 6x$
 $y' = \operatorname{sech}^2 6x \cdot \frac{d}{dx}(6x)$
 $= 6 \operatorname{sech}^2 6x$

2. $y = \cosh x^4$
 $y' = \sinh x^4 (x^4)'$
 $= 4x^3 \sinh x^4$

3. $y = \sinh^2 x$
 $y' = 2 \sinh x \cdot \frac{d}{dx}(\sinh x)$
 $= 2 \sinh x \cosh x$

4. $y = \ln(\cosh x)$
 $y' = \frac{\sinh x}{\cosh x} = \tanh x$

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5. $y = \sinh(\cosh x)$
 $y' = \cosh(\cosh x) \cdot \frac{d}{dx}(\cosh x) = \sinh x \cdot \cosh(\cosh x)$

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6. $y = e^{\cosh 3x}$
 $y' = e^{\cosh 3x} (\cosh 3x)' = (3 \sinh 3x) e^{\cosh 3x}$

7. $y = \cosh(\ln x)$
 $= \frac{1}{2} [e^{\ln x} + e^{-\ln x}] = \frac{1}{2} [x + \frac{1}{x}]$
 $y' = \frac{1}{2} [1 - \frac{1}{x^2}] = \frac{x^2 - 1}{2x^2}$

3.9c Hyperbolic Functions (Cont.)

Inverse Hyperbolic Functions

Definition $y = \sinh^{-1} x \Leftrightarrow \sinh y = x$
 $y = \tanh^{-1} x \Leftrightarrow \tanh y = x$

Derivatives

1. $y = \sinh^{-1} x \Rightarrow \sinh y = x$
 $(\cosh y) \frac{dy}{dx} = \frac{dx}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{\cosh x} = \frac{1}{\sqrt{1 + \sinh^2 x}} = \frac{1}{\sqrt{1 + x^2}}$$

2. $y = \tanh^{-1} x \Rightarrow \tanh y = x$
 $(\operatorname{sech}^2 y) \frac{dy}{dx} = \frac{dx}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y} = \frac{1}{1 - \tanh^2 y} = \frac{1}{1 - x^2}$$

Summary:

$$1. \frac{d}{dx} \sinh^{-1} u = \frac{1}{\sqrt{1+u^2}} \frac{du}{dx}$$

$$2. \frac{d}{dx} \tanh^{-1} u = \frac{1}{1-u^2} \frac{du}{dx}$$

Examples

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1. $y = \tanh^{-1} \sqrt{x} \quad y' = \frac{1}{1-(\sqrt{x})^2} (\sqrt{x})' = \frac{1}{2\sqrt{x}(1-x)}$

2. $y = x \tanh^{-1} x + \ln \sqrt{1-x^2}$
 $y' = x \left(\frac{1}{1-x^2} \right) + \tanh^{-1} x + \frac{1}{2} \left(\frac{1}{1-x^2} \right) (-2x)$
 $= \frac{x}{1-x^2} + \tanh^{-1} x - \frac{x}{1-x^2} = \tanh^{-1} x$

Related Rates

3.10 Related Rates

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3.1 Related Rates

Introduction

Related Rates problems involve variables that change with respect to time.


To solve related rate problems, try the following strategy.

- 1) Draw a picture. Assign variables.
- 2) Write out the given data. Identify the unknown.
- 3) Write a relation between the appropriate variables
- 4) Differentiate with respect to time.
- 5) Solve for the unknown variable. Plug in the data.
- 6) Read the problem again. Did you answer the question?

Geometry Formulas

Examples

- 1) Balloon
- 2) Cone
- 3) Boat



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Helpful Formulas

- 1) PYTHAGORAS' THEOREM
- 2) AREAS AND VOLUMES

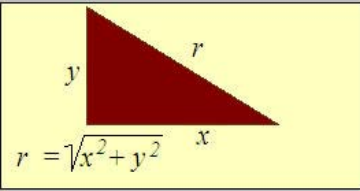
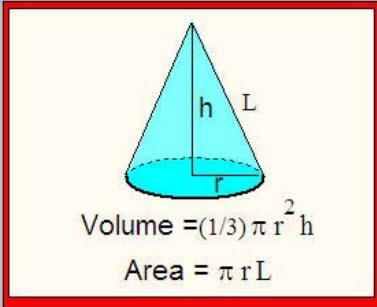
a) Box

b) Cylinder

c) Sphere


d) Cone

e) Pyramid


$$r = \sqrt{x^2 + y^2}$$


Volume = $(1/3) \pi r^2 h$

Area = $\pi r L$



3.10b Related Rates (Cont.)

Ch3_Calc I

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Helpful Formulas

1) PYTHAGORAS' THEOREM

2) AREAS AND VOLUMES

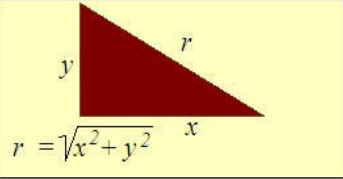
a) Box

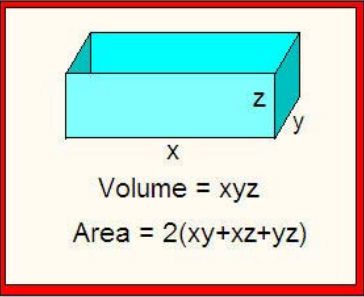
b) Cylinder

c) Sphere

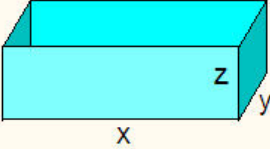
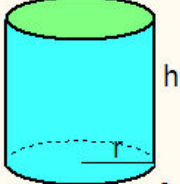
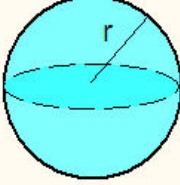
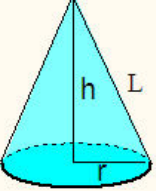
d) Cone

e) Pyramid



$$r = \sqrt{x^2 + y^2}$$


Volume = xyz
Area = $2(xy+xz+yz)$

 <p>Volume = xyz Area = $2(xy+xz+yz)$</p>	 <p>Volume = $\pi r^2 h$ Area = $2 \pi r h$</p>
 <p>Volume = $(4/3) \pi r^3$ Area = $4 \pi r^2$</p>	 <p>Volume = $(1/3) \pi r^2 h$ Area = $\pi r L$</p>

3.10b Related Rates (Cont.)

Ch3_Calc I
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Example 1. Balloon Rising

A balloon rising straight up. The distance r from the balloon to a range finder located 400 ft from the point of lift off is increasing at a rate of 150 ft/min. How fast is the balloon rising when $r = 500$ ft?

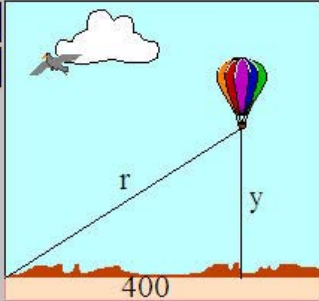
Solution

$$r^2 = y^2 + 400^2$$

$$\frac{d}{dt}(r^2) = \frac{d}{dt}(y^2 + 400^2)$$

$$2r \frac{dr}{dt} = 2y \frac{dy}{dt}$$

Procedure



Animate

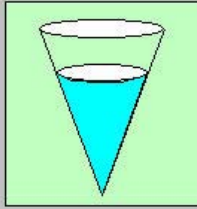
$$\frac{dy}{dt} = \frac{r}{y} \cdot \frac{dr}{dt}$$

$$\left. \frac{dy}{dt} \right|_{r=500} = \frac{500}{400} \cdot 150 = 187.5$$

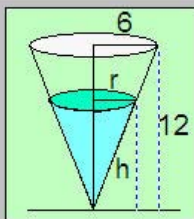
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Example 2. Filling a Cone

Water is pouring into a conical cistern at the rate of 8 cu. m /min. If the height of the cistern is 12 m and the radius of the circular opening is 6 m, how fast is the water level raising when the water is 4 m deep.



Empty



Fill

Solution

Given: $\frac{dV}{dt} = 8$ $h = 4$

$$\frac{r}{h} = \frac{6}{12}$$

$$r = \frac{h}{2}$$

$$V = \frac{1}{3} \pi r^2 h$$

$$V = \frac{1}{12} \pi h^3$$

$$\frac{dV}{dt} = \frac{4}{\pi h^2} \frac{dV}{dt}$$

$$\frac{dh}{dt} = \frac{4}{\pi 4^2} 8$$

$$\frac{dh}{dt} = \frac{2}{\pi} \sim 0.637$$

3.10c Related Rates (Cont.)

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Example 2. Speed of a Boat

The angle of depression to the bow of a boat from a 25 m tower is 0.4 rad when the boat is 150 m away. If the angle is changing at 0.02 rad/s, what is the speed of the boat?

Solution

$$\frac{d\theta}{dt} = 0.02 \quad x=150$$

$$\frac{x}{25} = \cot \theta \quad x = 25 \cot \theta$$

$$\frac{dx}{dt} = -25 \csc^2 \theta \frac{d\theta}{dt}$$

When $x=150$, $\csc \theta = \sqrt{37}$

$$\frac{dx}{dt} = -25(37)(0.02)$$

$$\frac{dx}{dt} = 18.5$$

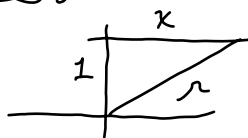
Animate

Examples

7

1. A plane flying horizontally at an altitude of 1 mile and a speed of 500 mph passes directly over a radar station. Find the rate at which the distance to the station is increasing when it is 2 miles away from the station.

Sol



$$1 + x^2 = r^2 \quad \frac{dr}{dt} = 500 \quad x=2 \quad \frac{dr}{dt} = ?$$

$$\frac{d}{dt}(1 + x^2) = \frac{d}{dt} r^2$$

$$2x \frac{dx}{dt} = 2r \frac{dr}{dt} \quad \frac{dr}{dt} = \frac{x}{r} \frac{dx}{dt}$$

$$\left. \frac{dr}{dt} \right|_{x=2} = \frac{2}{\sqrt{5}} \cdot 500 \approx 447.21 \text{ mph}$$

3.10d Related Rates (Cont.)

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2. If a snowball melts so that its surface area decreases at a rate of $1 \text{ cm}^2/\text{min}$, find the rate at which the diameter decreases when the diameter is 10 cm .

Sol:



$$S = 4\pi r^2 = 4\pi \left(\frac{D}{2}\right)^2 \quad D=10, \quad \frac{dS}{dt} = -1$$

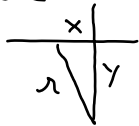
$$S = \pi D^2$$

$$\frac{dS}{dt} = 2\pi D \frac{dD}{dt} \quad \left. \frac{dD}{dt} \right|_{D=10} = \frac{1}{2\pi D} \left. \frac{dS}{dt} \right|_{D=10} = \frac{-1}{20\pi}$$

11

4. Two cars start moving from the same point. One travels south at 60 mph and the other travels east at 25 mph . At what rate is the distance between the cars increasing two hours later?

Sol



$$\frac{dy}{dt} = 60 \quad \text{At } t=2 \quad y = 120$$

$$x = 50$$

$$\frac{dr}{dt} = ?$$

$$\frac{dx}{dt} = 25$$

$$\therefore r = 130$$

$$x^2 + y^2 = r^2 \Rightarrow 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2r \frac{dr}{dt}$$

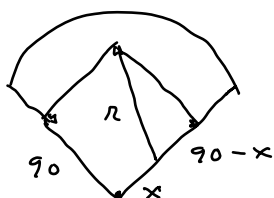
$$\left. \frac{dr}{dt} \right|_{t=2} = \frac{1}{r} \left[x \frac{dx}{dt} + y \frac{dy}{dt} \right] \Big|_{t=2} = \frac{1}{130} (50 \cdot 25 + 120 \cdot 60)$$

$$= 65 \text{ mph}$$

3.10e Related Rates (Cont.)

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5. A baseball diamond is a square with side 90 ft. A batter hits the ball and runs towards first base at 24 ft/s. How fast is his distance from second base decreasing when he is halfway to first base?

Sol

$$(90-x)^2 + 90^2 = r^2$$

$$x = 45 \text{ ft}$$

$$2(90-x)\left(-\frac{dx}{dt}\right) = 2r \frac{dr}{dt}$$

$$\frac{dx}{dt} = 24 \text{ ft/s}$$

$$\left. \frac{dr}{dt} \right|_{x=45} = -\frac{1}{r} (90-x) \left. \frac{dx}{dt} \right|_{x=45}$$

$$= -\frac{1}{\sqrt{45^2 + 90^2}} (90-45) \cdot 24$$

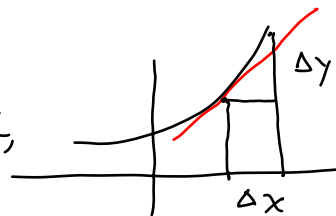
$$= -\frac{1}{45\sqrt{1+2^2}} (45)(24) = -\frac{24}{\sqrt{5}}$$

Differentials

3.11 Differentials

Let $y = f(x)$

As we have already learned, the derivative of y w.r.t x is given by the limit



$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

If Δx is small then we can approximate $f'(x)$ by

$$f'(x) \approx \frac{\Delta y}{\Delta x} \Rightarrow \Delta y \approx f'(x) \Delta x$$

The smaller Δx the better the approximation. Thus, at the infinitesimal level, we are lead to the definition

$$\boxed{dy = f'(x) dx} \quad dx \text{ is called the differential of } x$$

Differentials are useful for:

1. Approximations
2. Justification of the chain rule
3. Implicit differentiation
4. Linearization.

Examples

1. $y = x^2$ $dy = 2x dx$
2. $y = \sin x$ $dy = \cos x dx$
3. $y = e^x$ $dy = e^x dx$
4. $y = x e^x$ $dy = (x e^x + e^x) dx$

3.11a Differentials (Cont.)

Approximations

Ex: Find an approximation to $\sqrt{101}$ using differentials

Sol Let $y = \sqrt{x}$ $x = 100$ $\Delta x = 1$ $y = 10$
 $\Delta y = \frac{1}{2\sqrt{x}} \Delta x = \frac{1}{2\sqrt{100}} \cdot 1 = \frac{1}{20} = 0.05$

$$\sqrt{101} \approx y + \Delta y = 10 + 0.05 \qquad \sqrt{101} \approx 10.05$$

Chain Rule

Let $y = f(u)$ Then $dy = f'(u) du$ (Def.)
 $u = g(x)$ $\frac{dy}{dx} = f'(u) \frac{du}{dx}$ $f'(u) = \frac{dy}{du}$

$$\therefore \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

We can rewrite the derivative formulas of basic functions using differentials

Example 1. $\frac{d}{dx} e^u = e^u \frac{du}{dx} \Leftrightarrow d(e^u) = e^u du$

2. $\frac{d}{dx} \sin u = \cos u \frac{du}{dx} \Leftrightarrow d(\sin u) = \cos u du$

3. $\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx} \Leftrightarrow d(\ln u) = \frac{1}{u} du$

4. $d(uv) = u dv + v du$ etc.

3.11b Differentials (Cont.)

Implicit Differentiation

Examples

$$\begin{aligned}
 1. \quad x^2 + y^2 &= 1 \\
 d(x^2 + y^2) &= d(1) \\
 2x dx + 2y dy &= 0 \\
 2y dy &= -2x dx \\
 dy &= -\frac{x}{y} dx \\
 \frac{dy}{dx} &= -\frac{x}{y}
 \end{aligned}$$

$$\begin{aligned}
 2. \quad x^3 + xy &= 3 \\
 d(x^3 + xy) &= 0 \\
 3x^2 dx + x dy + y dx &= 0 \\
 x dy &= -(3x^2 + y) dx \\
 dy &= -\frac{1}{x}(3x^2 + y) dx \\
 \frac{dy}{dx} &= -\frac{(3x^2 + y)}{x}
 \end{aligned}$$

Linearization

The linearization of a function at a point is the equation of the tangent line at that point

Examples

1. Find the linearization of $x^3 + y^3 = 2$ at the point $(1, 1)$

$$\begin{aligned}
 \text{Sol: } d(x^3 + y^3) &= 0 \\
 3x^2 dx + 3y^2 dy &= 0 \\
 y^2 dy &= -x^2 dx \\
 dy &= -\frac{x^2}{y^2} dx
 \end{aligned}$$

$$\begin{aligned}
 \text{At } (1, 1) \\
 \Delta y &= y - 1 \\
 \Delta x &= x - 1
 \end{aligned}$$



$$y - 1 = -\left(\frac{1^2}{1^2}\right)(x - 1) \Rightarrow y - 1 = -x + 1 \\
 y = x + 2$$

Review

Differentiation Rules Calculus I

General Formulas

1. $\frac{d}{dx}c = 0$.
2. $\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$.
3. $\frac{d}{dx}(u - v) = \frac{du}{dx} - \frac{dv}{dx}$.
4. $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$.
5. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$.
6. $y = f(u)$, $u = u(x)$, $\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$.

Basic Functions

1. $\frac{d}{dx}e^u = e^u \frac{du}{dx}$.
2. $\frac{d}{dx} \ln |u| = \frac{1}{u} \frac{du}{dx}$.
3. $\frac{d}{dx} \sin u = \cos u \frac{du}{dx}$.
4. $\frac{d}{dx} \cos u = -\sin u \frac{du}{dx}$.
5. $\frac{d}{dx} \tan u = \sec^2 u \frac{du}{dx}$.
6. $\frac{d}{dx} \sec u = \sec u \tan u \frac{du}{dx}$.
7. $\frac{d}{dx} \cot u = -\csc^2 u \frac{du}{dx}$.
8. $\frac{d}{dx} \csc u = -\csc u \cot u \frac{du}{dx}$.
9. $\frac{d}{dx} \sinh u = \cosh u \frac{du}{dx}$.
10. $\frac{d}{dx} \cosh u = \sinh u \frac{du}{dx}$.
11. $\frac{d}{dx} \tanh u = \operatorname{sech}^2 u \frac{du}{dx}$.
12. $\frac{d}{dx} \sin^{-1} u = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$.
13. $\frac{d}{dx} \tan^{-1} u = \frac{1}{1+u^2} \frac{du}{dx}$.
14. $\frac{d}{dx} \sec^{-1} u = \frac{1}{u\sqrt{u^2-1}} \frac{du}{dx}$.
15. $\frac{d}{dx} \sinh^{-1} u = \frac{1}{\sqrt{1+u^2}} \frac{du}{dx}$.
16. $\frac{d}{dx} \tanh^{-1} u = \frac{1}{1-u^2} \frac{du}{dx}$.