Polynomials and Exponentials

3.1 Polynomials and Exponentials

We would like to get formular for the derivative of x^n.

Recall: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$n = 0$ $\quad f(x) = x^0 = 1$

$\quad f'(x) = 0$

$n = 1$ $\quad f(x) = x^1 = x$

$\quad f'(x) = 1$ (Slope = 1)

$n = 2$ $\quad f(x) = x^2$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} [x^2 + 2hx + h^2 - x^2]$$

$$= \lim_{h \to 0} \frac{1}{h} [2hx + h^2] = \lim_{h \to 0} [2x + h]$$

$$= 2x$$

$n = 3$ $\quad f(x) = x^3$

$$f'(x) = \lim_{h \to 0} \frac{1}{h} [(x+h)^3 - x^3]$$

$$= \lim_{h \to 0} \frac{1}{h} [x^3 + 3x^2h + 3xh^2 + h^3 - x^3]$$

$$= \lim_{h \to 0} \frac{1}{h} [3x^2h + 3xh^2 + h^3]$$

$$= \lim_{h \to 0} [3x^2 + 3xh + h^2]$$

$$= 3x^2$$

Summary

$f(x)$	$f'(x)$
x^0	0
x^1	1
x^2	$2x$
x^3	$3x^2$

$f(x) = x^n$

First, we need to review Newton's binomial formula

$(x+h)^1 = 1x + 1h$

$(x+h)^2 = 1x^2 + 2xh + 1h^2$

$(x+h)^3 = 1x^3 + 3x^2h + 3xh^2 + 1h^3$

n	Coeff			
1		1	1	
2	1	2	1	
3	1	3	3	1

Pattern:
1. Powers of x decrease by 1
2. Power of h increase by 1
3. Coefficients given by Pascal's Triangle

Formula for the coefficients

Let n = power of the binomial

k = power of x in the expansion

$\binom{n}{k}$ = Coefficient of x^k

Then $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ $\quad n! = n(n-1)(n-2)\cdots 2\cdot 1$

$0! = 1$

Example $n = 3$

$\binom{3}{0} = \dfrac{3!}{0!(3-0)!} = 1$

$\binom{3}{1} = \dfrac{3!}{1!(3-1)!} = \dfrac{3!}{2!} = \dfrac{3\cdot 2\cdot 1}{2\cdot 1} = 3$

$\binom{3}{2} = \dfrac{3!}{2!(3-2)!} = \dfrac{3!}{2!} = \dfrac{3\cdot 2\cdot 1}{2\cdot 1} = 3$

$\binom{3}{3} = \dfrac{3!}{3!(3-3)!} = \dfrac{3!}{3!0!} = \dfrac{3!}{3!} = 1$

So: $(x+h)^3 = \binom{3}{0}x^3h^0 + \binom{3}{1}x^2h^1 + \binom{3}{2}x^1h^2 + \binom{3}{3}x^0h^3$

3.1b Polynomials (Cont.)

$$(x+h)^n = \binom{n}{0} x^n h^0 + \binom{n}{1} x^{n-1} h^1 + \binom{n}{2} x^{n-2} h^2 + \binom{n}{3} x^{n-3} h^3 + \cdots$$

$$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{n!} = 1$$

$$\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{n!}{(n-1)!} = \frac{n(n-1)(n-2)\cdots 2\cdot 1}{(n-1)(n-2)\cdots 2\cdot 1} = n$$

$$\therefore (x+h)^n = x^n h^0 + n x^{n-1} h^1 + \binom{n}{2} x^{n-2} h^2 + \cdots\cdots$$

$$(x+h)^n - x^n = n x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \binom{n}{3} x^{n-3} h^3 + \cdots$$

$$\frac{(x+h)^n - x^n}{h} = n x^{n-1} + \binom{n}{2} x^{n-2} h + \binom{n}{3} x^{n-3} h^2 + \cdots$$

$$\frac{d}{dx} x^n = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0} \left[n x^{n-1} + \binom{n}{2} x^{n-2} h + \binom{n}{3} x^{n-3} h^2 + \cdots \right]$$

$$= n x^{n-1}$$

Formula: $\boxed{\frac{d}{dx}(x^n) = n x^{n-1}}$ Power Rule.

Examples

1) $\frac{d}{dx} x^5 = 5x^4$

2) $\frac{d}{dx} x^{10} = 10x^9$

3) $\frac{d}{dx} 3x^7 = 3\frac{d}{dx} x^7 = 3(7x^6) = 21x^6$

4) $\frac{d}{dx}(x^4 + x^2) = 4x^3 + 2x$

5) $\frac{d}{dx}\left[3x^5 + 4x^2 + 5\right] = 5(3)x^4 + 2(4)x + 0$
$= 15x^4 + 8x$

Note. The power rule is true for any n, not just integers!

Ex $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx} x^{-1} = -1x^{-2} = -\frac{1}{x^2}$

3.1c Polynomials (Cont.)

Algebra

1) $x^{-n} = \frac{1}{x^n}$

2) $x^{\frac{1}{n}} = \sqrt[n]{x}$

3) $x^{\frac{m}{n}} = \sqrt[n]{x^m}$

Ex $\frac{d}{dx}\sqrt{x} = \frac{d}{dx} x^{\frac{1}{2}} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$

This one is worth remembering

$$\boxed{\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}}$$

Ex $\frac{d}{dx} x\sqrt{x} = \frac{d}{dx} x \cdot x^{1/2} = \frac{d}{dx} x^{3/2} = \frac{3}{2} x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$

Ex $\frac{d}{dx} x^{\frac{5}{2}} = \frac{5}{2} x^{\frac{3}{2}}$

Ex $\frac{d}{dx} \frac{x^2 - 3x}{\sqrt{x}} = \frac{d}{dx}\left[x^{2-\frac{1}{2}} - 3x^{1-\frac{1}{2}}\right]$

$= \frac{d}{dx}\left[x^{3/2} - 3^{1/2}\right]$

$= \frac{3}{2} x^{\frac{1}{2}} - \frac{3}{2} x^{-1/2}$

$= \frac{3}{2}\sqrt{x} - \frac{3}{2\sqrt{x}}$

Ex $\frac{d}{dx} x[x^2 - 1]^2 = \frac{d}{dx} x(x^4 - 2x^2 + 1)$

$= \frac{d}{dx}(x^5 - 2x^3 + x) = 5x^4 - 6x^2 + 1$

Let $f(x) = a^x \quad a > 0$

$$f'(x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x a^h - a^x}{h}$$

$$= \lim_{h \to 0} a^x \frac{[a^h - 1]}{h}$$

$$= a^x \lim_{h \to 0} \left[\frac{a^h - 1}{h}\right]$$

← For now, we can use a graphing calculator to evaluate the limit numerically.

Call this limit $\ln a$

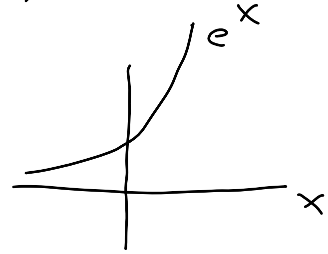
$a = 2 \quad \lim_{h \to 0} \left[\frac{2^h - 1}{h}\right] = 0.693 \quad = \ln 2$

$a = 3 \quad \lim_{h \to 0} \left[\frac{3^h - 1}{h}\right] = 1.0986 \quad = \ln 3$

$a = 4 \quad \lim_{h \to 0} \left[\frac{4^h - 1}{h}\right] = 1.3863 \quad = \ln 4$

For some number between 2 and 3, there is a unique number for which the limit above is equal to 1. This number is called e (for Euler.)

$e \doteq 2.71828 \qquad \ln e = 1$



Conclusion:

$f(x) = a^x$
$f'(x) = a^x \ln a$

$f(x) = e^x$
$f'(x) = e^x$

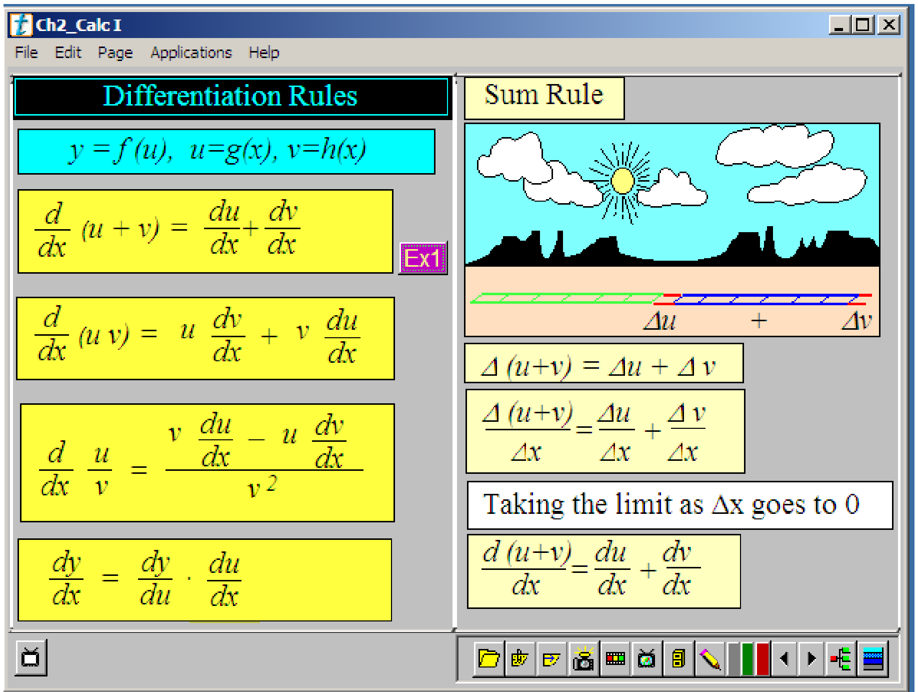
The function $f(x) = e^x$ is the only function whose derivative is itself

Ex: $\frac{d}{dx} 2^x = 2^x \ln 2 \qquad \frac{d}{dx} 5e^x = 5e^x$

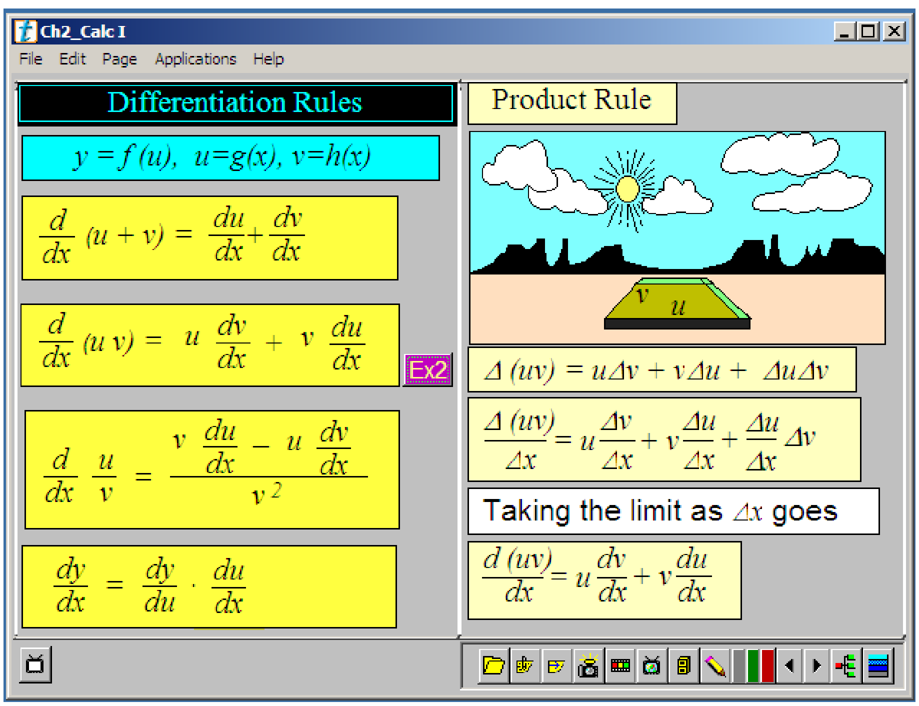
$\frac{d}{dx} 7^x = 7^x \ln 7 \qquad \frac{d}{dx} \frac{e^x}{2} = \frac{e^x}{2}$

Product and Quotient Rules

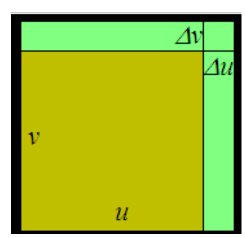
3.2 Sum and Product Rules



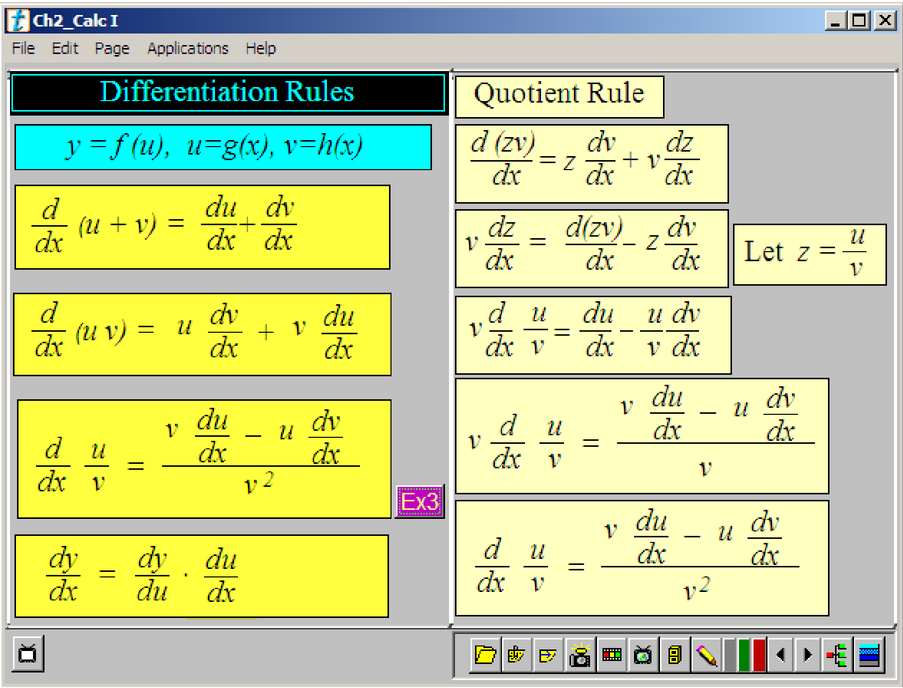
Visual "proof" of Sum rule



Visual "proof" of Product rule.



3.2a Quotient Rule



Proof of Quotient Rule

Ex 1: $y = x^2 e^x$

$y' = x^2 \frac{d}{dx} e^x + e^x \frac{d}{dx}(x^2)$

$= x^2 e^x + e^x 2x$

$= e^x(x^2 + 2x)$

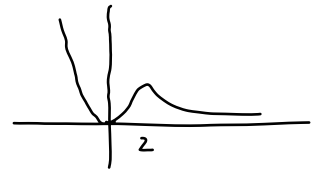
Ex 2: $y = \frac{e^x + 1}{e^x - 1}$ $\quad y' = \frac{(e^x-1)(e^x+1)' - (e^x+1)'(e^x-1)}{(e^x-1)^2}$

$y' = \frac{(e^x-1)e^x - (e^x-1)e^x}{(e^x-1)^2}$

$y' = \frac{e^x[(e^x-1)-(e^x-1)]}{(e^x-1)^2} = \frac{-2e^x}{(e^x-1)^2}$

Ex 3 $y = \frac{x}{x+1}$

$y' = \frac{(x+1)(x)' - x(x+1)'}{(x+1)^2} = \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)^2}$



Ex $y = x^2 e^{-x}$ Find y'

$y = \dfrac{x^2}{e^x}$

$y' = \dfrac{(e^x)(x^2)' - x^2(e^x)'}{e^{2x}}$

$= \dfrac{e^x(2x) - x^2 e^x}{e^{2x}} = \dfrac{e^x[2x - x^2]}{e^{2x}} = x(2-x)e^{-x}$

Note that the function has 0-slope at $x = 0$ and $x = 2$

Ex $y = \dfrac{x}{x + \frac{c}{x}}$. Multiply top and bottom by x

$y = \dfrac{x^2}{x^2 + c}$

$y' = \dfrac{(x^2+c)(x^2)' - x^2(x^2+c)'}{(x^2+c)^2} = \dfrac{(x^2+c)(2x) - x^2(2x)}{(x^2+c)^2}$

$y' = \dfrac{2x[(x^2+c) - x^2]}{(x^2+c)^2} = \dfrac{2xc}{(x^2+c)^2}$

Ex $y = \dfrac{x^2 + 4x + 3}{\sqrt{x}}$. Better to avoid the the quotient rule

$y = \dfrac{x^2 + 4x + 3}{x^{1/2}}$

$= x^{3/2} + 4x^{1/2} + 3x^{-1/2}$

$y' = \frac{3}{2}x^{1/2} + \frac{4}{2}x^{-1/2} - \frac{3}{2}x^{-3/2} = \frac{3}{2}\sqrt{x} + \frac{2}{\sqrt{x}} - \frac{3}{2x\sqrt{x}}$

$y' = \dfrac{3x^2 + 4x - 3}{2x\sqrt{x}}$ ← Common Denom.

Rates of Change in the Sciences

3.3 Rates of Change in the Sciences

Physics:

$x = x(t)$ Position

$\frac{dx}{dt} = v(t)$ Velocity

$\frac{dv}{dt} = a(t)$ Acceleration.

Uniform Motion.

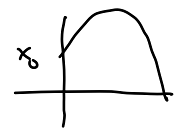
A particle is thrown upwards at the surface of the Earth with initial velocity v_0 and initial height x_0.

Then

$$a = \frac{dv}{dt} = -g \qquad g = 9.80\ m/s^2 = 32\ ft/s^2$$

$$v = -gt + v_0 \qquad \left(\frac{dv}{dt} = -g, \quad v(0) = -g(0) + v_0 = v_0 \right)$$

$$x = -\frac{1}{2} g t^2 + v_0 t + x_0 \qquad \left(\frac{dx}{dt} = v \qquad x(0) = x_0 \right)$$



In general, a particle moving with constant acceleration (a = const) satisfies the equation

$$\begin{cases} v = at + v_0 & v(0) = v_0 \\ x = \frac{1}{2} a t^2 + v_0 t + x_0 & x(0) = x_0 \end{cases}$$

The velocity is a linear function

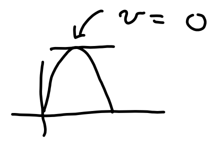
The path of the particle is parabolic

Example: A ball is thrown upwards from the ground with a velocity of 80 ft/s. Find
a) The maximum height
b) The velocity when the height is 96 ft

Sol:

$x_0 = 0$, $v_0 = 80$

$$\begin{cases} x = -\frac{1}{2}(32)t^2 + 80t = -16t^2 + 80t \\ v = -32t + 80 \end{cases}$$



a) Maximum height: set $v = 0$

$-32t + 80 = 0$

$t = \frac{80}{32} = \frac{10}{4} = \frac{5}{2} = 2.5\text{ s}$

b) $x = -16t^2 + 80t = 96 \Rightarrow t^2 - 5t + 6 = 0$

$16t^2 - 80t + 96 = 0$ $\quad (t-3)(t-2) = 0 \quad t = 2, 3$

$v(2) = -32(2) + 80 = 16$ up

$v(3) = -32(3) + 80 = -16$ Down

Ex: The position of a particle is given by
$s = t^3 - 4.5t^2 - 7t \quad t \geq 0$
When does the particle reach a velocity of 5 m/s

Sol $v = 3t^2 - 9t - 7 = 5$

$3t^2 - 9t - 12 = 0$

$t^2 - 3t - 4 = 0$

$(t-4)(t+1) = 0 \quad t = 4$

Ex The charge Q in a wire is given by
$Q = t^3 - 2t^2 + 6t + 2$ (Coulombs)
Find the current I when $t = 1$ sec

Sol $I = \frac{dQ}{dt} = 3t^2 - 4t + 6$

$I(1) = 3 - 4 + 6 = 5$ amps

Chemistry

Rates of Chemical Reactions

Example: Consider the reaction

$$A + B \rightarrow C$$

Let [A], [B] and [C] be the concentrations

Suppose that:

$[A] = [B] = a$ moles/L and

$[C] = \dfrac{a^2 k t}{akt + 1}$ $\quad k = \text{const}$

a) Find the rate of the reaction at time t

Sol

$$\frac{d}{dt}[C] = \frac{(akt+1)(a^2kt)' - a^2kt\,[akt+1]'}{(akt+1)^2}$$

$$= \frac{(akt+1)(a^2k) - a^2kt\,(ak)}{(akt+1)^2}$$

$$= \frac{a^2k\,[akt+1-akt]}{(akt+1)^2} = \frac{a^2k}{(akt+1)^2}$$

b) Let $x = [C]$

Show that $\dfrac{dx}{dt} = k(a-x)^2$

Sol

$$k(a-x)^2 = k\left[a - \frac{a^2kt}{akt+1}\right]^2 = k\left[\frac{a^2kt + a - a^2kt}{akt+1}\right]^2$$

$$= k\,\frac{a^2}{(akt+1)^2}$$

$$= \frac{dx}{dt}$$

c) What happens as $t \to \infty$

Sol $\displaystyle\lim_{t\to\infty}[C] = \lim_{t\to\infty}\frac{a^2kt}{akt+1} = a$

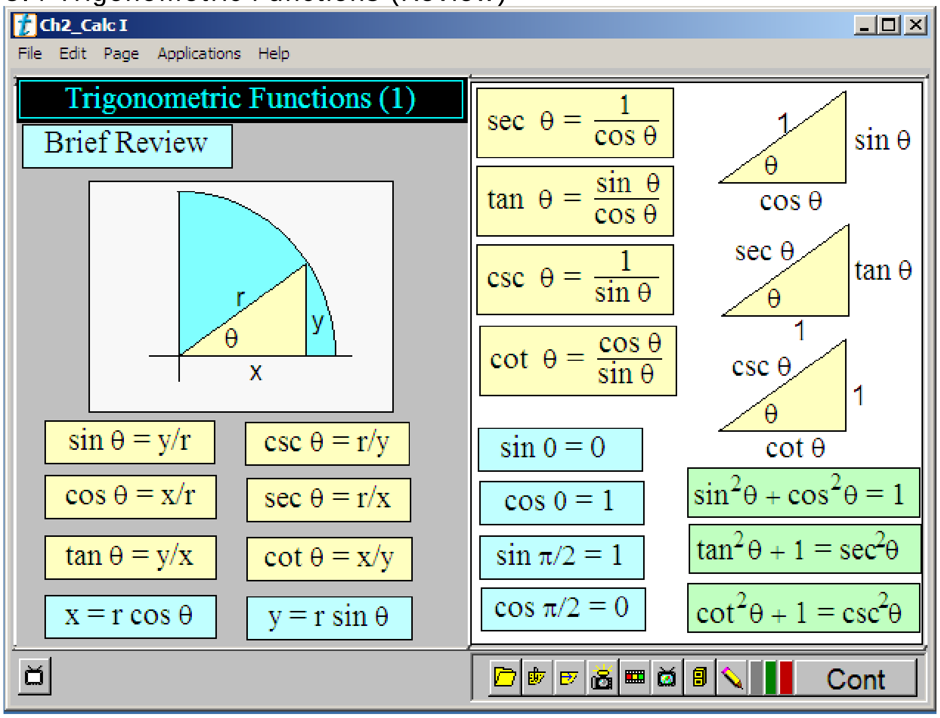
$$\lim_{t\to\infty}\frac{d[C]}{dt} = \lim_{t\to\infty}\frac{a^2k}{(akt+1)^2} = 0$$

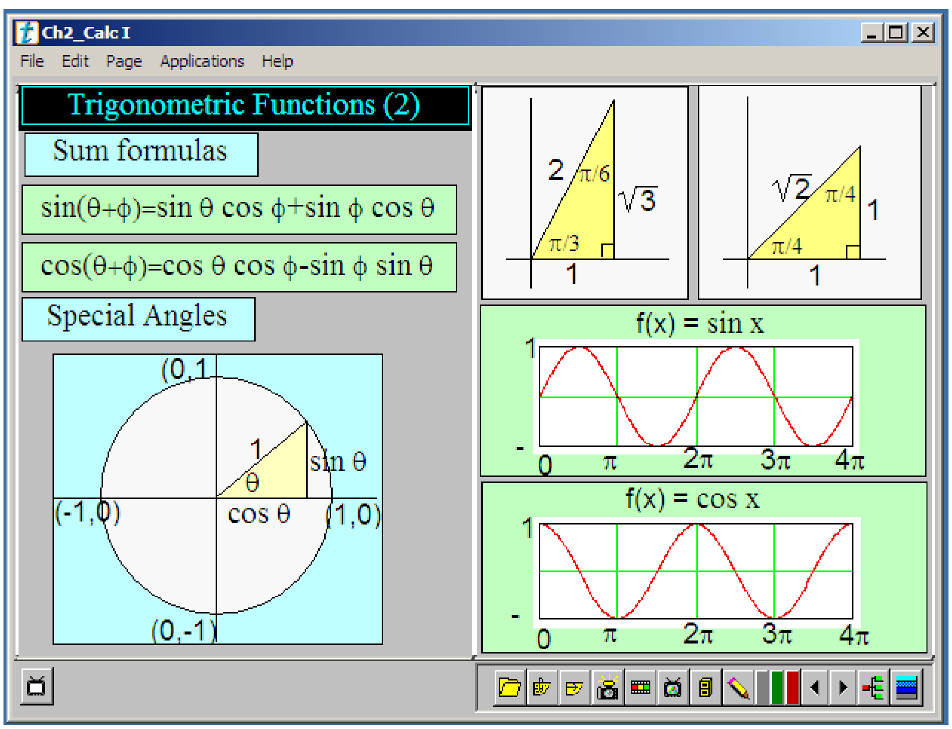
As $t \to \infty$ the number of C molecules approaches the number of A molecules and the reaction rate goes to zero.

Trigonometric Functions

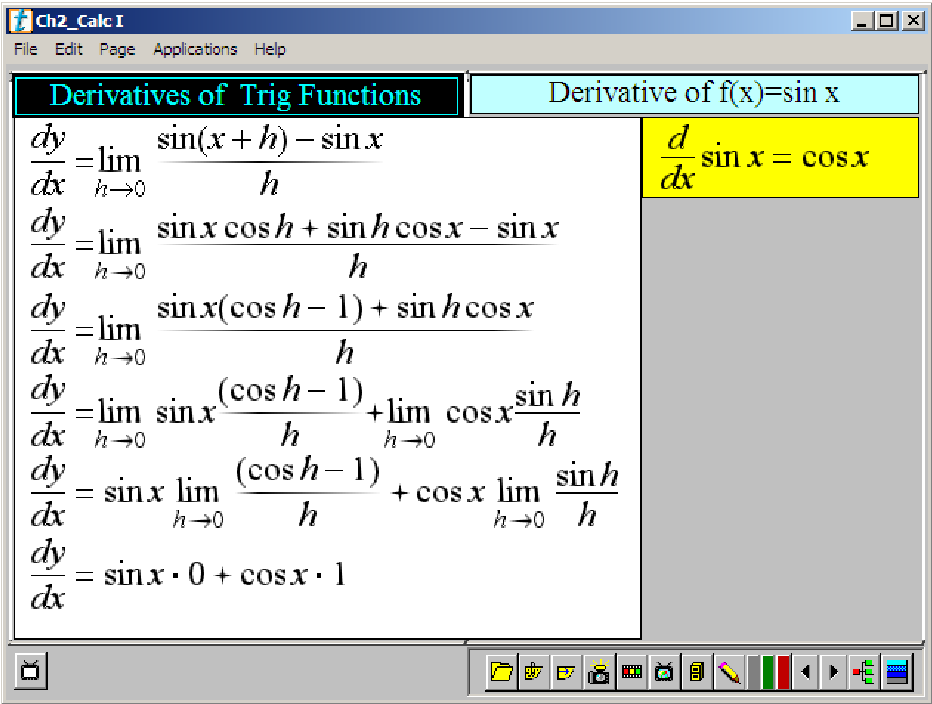
3.4 Trigonometric Functions (Review)

I.





3.4a Trigonometric Functions



We seek a formula for the derivative of $f(x) = \sin x$

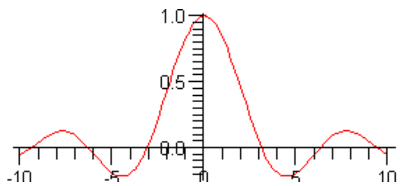
$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

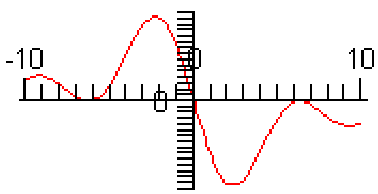
Use: $\sin(x+y) = \sin x \cos y + \sin y \cos x$

$$= \lim_{h\to 0} \frac{1}{h}[\sin x \cos h + \sin h \cos x - \sin x]$$

$$= \lim_{h\to 0} \frac{1}{h}[\sin x(\cos h - 1) + \sin h \cos x]$$

$$= (\sin x)\lim_{h\to 0}\left(\frac{\cos h - 1}{h}\right) + (\cos x)\lim_{h\to 0}\frac{\sin h}{h}$$





Using Maple

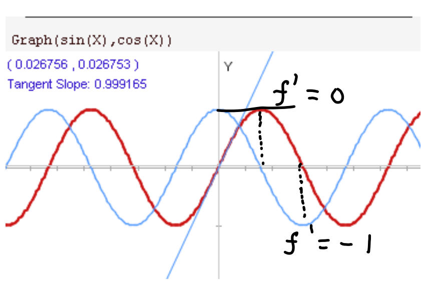
$$\lim_{h\to 0} \frac{\sin h}{h} = 1 \qquad \lim_{h\to 0}\frac{\cos h - 1}{h} = 0$$

Important limits to remember.

$$\therefore \frac{d}{dx}\sin x = \cos x$$

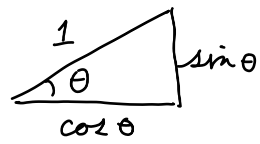
In a similar fashion one can show

$$\frac{d}{dx}\cos x = -\sin x$$



The formula for the derivative of $\sin x$ makes sense graphically. Notice that at $x = 0$ $\sin x$ has slope $= 1$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x\,(\sin x)' - \sin x\,(\cos x)'}{\cos^2 x}$$
$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$
$$= \frac{1}{\cos^2 x} = \sec^2 x$$



$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \frac{\cos x\cdot(0) - 1(-\sin x)}{\cos^2 x}$$
$$= \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}$$
$$= \tan x \sec x$$

Similarly: $\frac{d}{dx}\cot x = -\csc^2 x$

$\frac{d}{dx}\csc x = -\cot x \csc x$

Summary

1) $\frac{d}{dx}\sin x = \cos x$

2) $\frac{d}{dx}\cos x = -\sin x$

3) $\frac{d}{dx}\tan x = \sec^2 x$

4) $\frac{d}{dx}\csc x = -\csc x \cot x$

5) $\frac{d}{dx}\sec x = \sec x \tan x$

6) $\frac{d}{dx}\cot x = -\csc^2 x$

Derivatives of Trig Functions
$\frac{d}{dx}\sin x = \cos x$
$\frac{d}{dx}\cos x = -\sin x$
$\frac{d}{dx}\sec x = \sec x \tan x$
$\frac{d}{dx}\tan x = \sec^2 x$
$\frac{d}{dx}\cot x = -\csc^2 x$
$\frac{d}{dx}\csc x = -\csc x \cot x$

Examples: Differentiate

1) $f(x) = x \sin x$

$f'(x) = x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x)$

$= x \cos x + \sin x$

2) $f(x) = e^x \sin x$

$f'(x) = e^x (\sin x)' + \sin x (e^x)'$

$= e^x \cos x + \sin x \cdot e^x$

$= e^x (\cos x + \sin x)$

3) $y = \frac{\tan x}{x}$ $\quad y' = \frac{x(\tan x)' - \tan x (x)'}{x^2}$

$= \frac{1}{x^2} [x \sec^2 x - \tan x]$

4) $y = \csc x \cot x$ $\quad y' = \csc x (\cot x)' + \cot x (\csc x)'$

$= \csc x (-\csc^2 x) + \cot x (-\csc x \cot x)$

$= -\csc x [\csc^2 x + \cot^2 x]$

Evaluate the given limits

5) $\lim_{x \to 0} \frac{\sin 5x}{x} = \lim_{5x \to 0} 5 \cdot \frac{\sin 5x}{5x}$ $\quad$ Let $h = 5x$

$= 5 \lim_{h \to 0} \frac{\sin h}{h} = 5$

38 6) $\lim_{\theta \to 0} \frac{\cos\theta - 1}{\sin\theta} = \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} \cdot \lim_{\theta \to 0} \frac{\theta}{\sin\theta} = 0 \cdot \frac{1}{1} = 0$

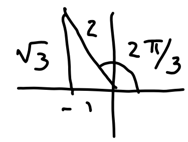
7) $\lim_{x \to 0} \frac{\sin 3x}{\sin 5x} = \lim_{x \to 0} \frac{5}{3} \frac{\sin 3x}{3x} \frac{5x}{\sin 5x} = \frac{5}{3} \cdot 1 \cdot \frac{1}{1} = \frac{5}{3}$

8) $\lim_{x \to 0} \frac{\tan x}{4x} = \lim_{x \to 0} \frac{\sin x}{x} \frac{1}{4 \cos x} = 1 \cdot \frac{1}{4} = \frac{1}{4}$

 – Questions

31) **Ex:** Let $x(t) = 8\sin t$. Find the velocity at $t = 2\pi/3$

Sol:



$$v(t) = x'(t) = 8\cos t$$

$$v\left(\frac{2\pi}{3}\right) = 8\cos\left(\frac{2\pi}{3}\right) = 8\left(-\frac{1}{2}\right) = -4$$

11) **Ex:** $f(x) = \dfrac{\sec\theta}{1+\sec\theta}$

$$f'(x) = \frac{(1+\sec\theta)(\sec\theta\tan\theta) - \sec\theta(\sec\theta\tan\theta)}{(1+\sec\theta)^2}$$

$$= \frac{\sec\theta\tan\theta\,[1+\sec\theta - \sec\theta]}{(1+\sec\theta)^2}$$

$$= \frac{\sec\theta\tan\theta}{(1+\sec\theta)^2}$$

Chain Rule

3.5 Chain Rule

Theorem: If $y = f(u)$ and $u = g(x)$, then

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$ – Newtons notation

or

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$ – Leibnitz notation

This is called the Chain rule factor

The chain rule is very powerful. It can be used to generalize the formulas for the derivatives of the elementary functions

Ex If $y = u^n$, $u = g(x)$ then $\frac{dy}{dx} = n u^{n-1}\frac{du}{dx}$

If $y = \sin u$, $u = g(x)$ then $\frac{dy}{dx} = \cos u \frac{du}{dx}$ etc.

Ex 1) $\frac{d}{dx}(x^3+2)^4 = 4(x^3+2)^3 \frac{d}{dx}(x^3+2) = 12x^2(x^3+2)^3$

2) $\frac{d}{dx}\sin(x^4) = \cos x^4 \cdot \frac{d}{dx}(x^4) = 4x^3 \cos x^4$

3) $\frac{d}{dx} e^{3x+2} = e^{3x+2}\frac{d}{dx}(3x+2) = 3e^{3x+2}$

4) $\frac{d}{dx}\cos(e^x) = -\sin(e^x)\frac{d}{dx}(e^x) = -e^x \sin(e^x)$

5) $\frac{d}{dx}\sec(5x) = \sec(5x)\tan(5x)\frac{d}{dx}(5x)$
$= 5\sec 5x \tan 5x$

6) $\frac{d}{dx}\sqrt{x^2+1} = \frac{1}{2}(x^2+1)^{-\frac{1}{2}}\frac{d}{dx}(x^2+1) = \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$

7) $\frac{d}{dx}\tan(\sin x) = \sec^2(\sin x) \cdot \frac{d}{dx}(\sin x)$
$= \cos x \cdot \sec^2(\sin x)$

8) $\frac{d}{dx} x e^{-x^2} = e^{-x^2}\frac{d}{dx}(x) + x e^{-x^2}\frac{d}{dx}(-x^2) = e^{-x^2}[1-2x^2]$

More Examples:

1) $\frac{d}{dx}\sin^3(x^2) = 3\sin^2(x^2)\,\frac{d}{dx}(\sin x^2)$

$= 3\sin^2(x^2)\cdot\cos(x^2)\cdot\frac{d}{dx}(x^2)$

$= 6x\sin^2(x^2)\cos(x^2)$

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2) $\frac{d}{dx}e^{x\cos x} = e^{x\cos x}\cdot\frac{d}{dx}(x\cos x)$

$= e^{x\cos x}[\cos x - x\sin x]$

3) $y = (2x+1)^3(3x-2)^5$

$y' = (2x+1)^3\frac{d}{dx}(3x-2)^5 + (3x-2)^5\frac{d}{dx}(2x+1)^3$

$= (2x+1)^3\cdot 5(3x-2)^4\cdot 3 + (3x-2)^5\cdot 3(2x+1)^2\cdot 2$

$= (2x+1)^2(3x-2)^4[(2x+1)(15) + (3x-2)6]$

$= (2x+1)^2(3x-2)^4[30x+15+9x-12]$

$= (2x+1)^2(3x-2)^4(39x-3)$

$= 3(2x+1)^2(3x-2)^4(13x-1)$

4) $y = \sqrt{x+\sqrt{x}}$

$y' = \frac{1}{2\sqrt{x+\sqrt{x}}}\cdot\frac{d}{dx}(x+\sqrt{x}) = \frac{1}{2\sqrt{x+\sqrt{x}}}\left(1+\frac{1}{2\sqrt{x}}\right)$

5) $y = \sin^2(\cos 5x)$

$y' = 2\sin(\cos 5x)\cdot\frac{d}{dx}(\cos 5x)$

$= 2\sin(\cos 5x)(-\sin 5x)\frac{d}{dx}(5x)$

$= -10\sin(\cos 5x)\sin 5x.$

3.5b Chain Rule

Problems

25) 1. $y = \sqrt{\dfrac{z-1}{z+1}}$ Note $\dfrac{d}{dx}\sqrt{u} = \dfrac{1}{2\sqrt{u}}\dfrac{du}{dx}$

$y' = \dfrac{1}{2\sqrt{\dfrac{z-1}{z+1}}} \dfrac{d}{dz} \dfrac{z-1}{z+1}$

$= \dfrac{1}{2}\sqrt{\dfrac{z+1}{z-1}} \dfrac{(z+1)(1)-(z-1)(1)}{(z+1)^2} = \dfrac{1}{2}\sqrt{\dfrac{z+1}{z-1}} \dfrac{2}{(z+1)^2}$

$= \dfrac{1}{(z+1)^2}\sqrt{\dfrac{z+1}{z-1}}$

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2. $y = \dfrac{r}{\sqrt{r^2+1}}$

$y' = \dfrac{\sqrt{r^2+1}\,(r)' - r\,(\sqrt{r^2+1})'}{r^2+1}$

$= \dfrac{\sqrt{r^2+1} - r\left(\dfrac{1}{2\sqrt{r^2+1}}\right)\cdot 2r}{r^2+1} =$

$= \left[\dfrac{\sqrt{r^2+1} - \dfrac{r^2}{\sqrt{r^2+1}}}{r^2+1}\right] \dfrac{\sqrt{r^2+1}}{\sqrt{r^2+1}} = \dfrac{r^2+1-r^2}{(r^2+1)^{3/2}}$

$= \dfrac{1}{(r^2+1)^{3/2}}$

35

3. $y = \sec^2 x + \tan^2 x$

$y' = 2\sec x\,(\sec x)' + 2\tan x\,(\tan x)'$

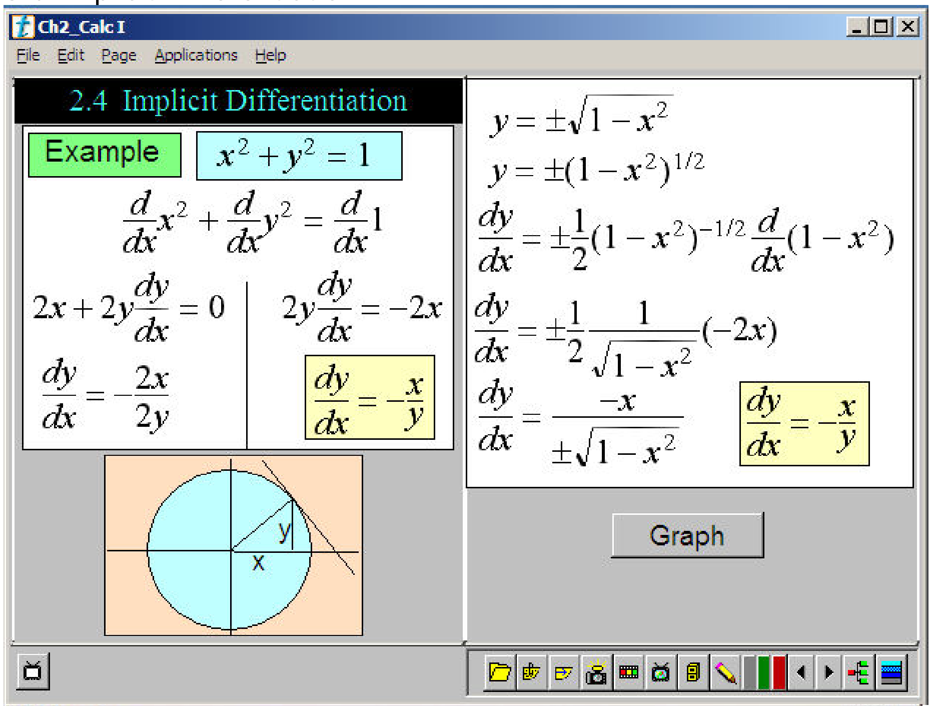
$= 2\sec x\,(\sec x \tan x) + 2\tan x\,(\sec^2 x)$

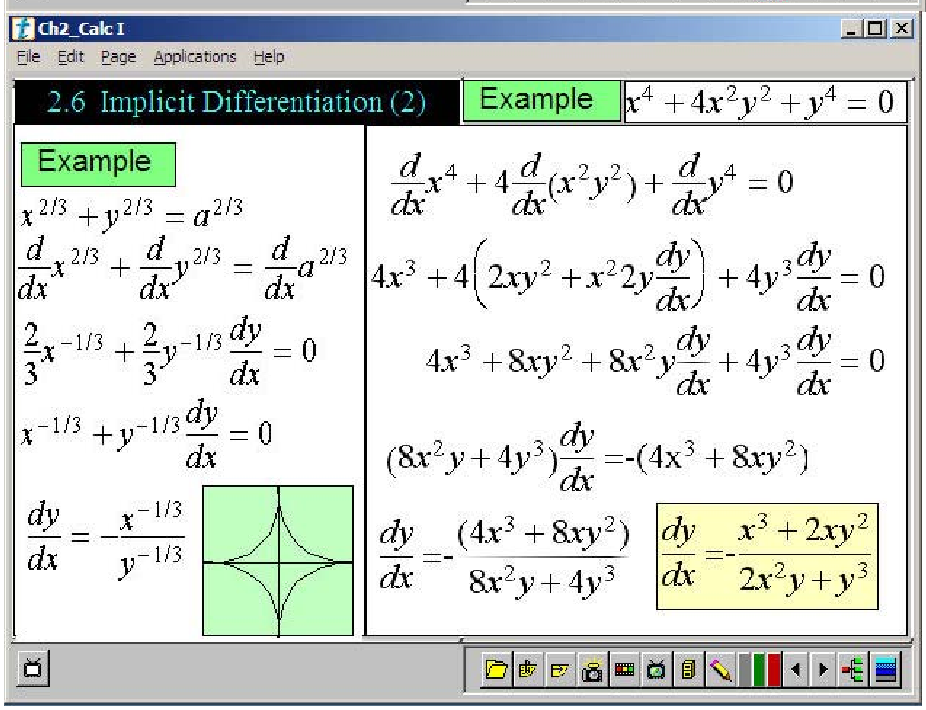
$= 2\sec^2 x\,[\tan x + \tan x]$

$= 4\sec^2 x \tan x$

Implicit Differentiation

3.6 Implicit Differentiation





Examples

9 1) $xy = \cot(xy)$

$\frac{d}{dx}(xy) = \frac{d}{dx}\cot(xy) \Rightarrow y + x\frac{dy}{dx} = -\csc^2(xy)\left[y + x\frac{dy}{dx}\right]$

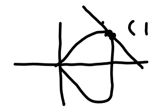
$y + xy' = -y\csc^2(xy) - xy'\csc^2(xy)$

$[x + x\csc^2(xy)]\,y' = -y - y\csc^2(xy)$

$y' = \frac{-y(1+\csc^2(xy))}{x(1+\csc^2(xy))} = \frac{-y}{x}$

2) Find the equation of the tangent line to

$y^2 = x^3(2-x)$ at the point $(1,1)$



Sol

$y^2 = 2x^3 - x^4$

$2y\frac{dy}{dx} = 6x^2 - 4x^3 = 2x^3(3x-4)$

$\frac{dy}{dx} = \frac{x^3(3x-4)}{y}$. $\left.\frac{dy}{dx}\right|_{(1,1)} = \frac{1(3-4)}{1} = -1 = m$

$y - 1 = -1(x-1)$

$y = -x + 1 + 1$ $\quad y = -x + 2$

13 3. $4\cos x\sin y = 1$

$4\cos x\frac{d}{dx}\sin y + 4\frac{d}{dx}(\cos x)\sin y = 0$

$4\cos x\cos y\frac{dy}{dx} - 4\sin x\sin y = 0$

$\cos x\cos y\frac{dy}{dx} = \sin x\sin y$

$\frac{dy}{dx} = \frac{\sin x\sin y}{\cos x\cos y} = \tan x\tan y$

15 4. $e^{x^2y} = x + y$

$e^{x^2y}\left(2xy + x^2\frac{dy}{dx}\right) = 1 + \frac{dy}{dx}$

$\left[x^2e^{x^2y} - 1\right]\frac{dy}{dx} = 1 - 2xye^{x^2y} \Rightarrow y' = \frac{1-2xye^{x^2y}}{x^2e^{x^2y} - 1}$

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5) $\sqrt{xy} = 1 + x^2 y$

$$\frac{1}{2\sqrt{xy}}[xy' + x] = x^2 y' + 2xy$$

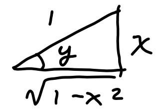
$$xy' + x = 2x^2\sqrt{xy}\,y' + 2xy \cdot 2\sqrt{xy}$$

$$(x - 2x^2\sqrt{xy})y' = 4xy\sqrt{xy} - x$$

$$y' = \frac{4xy\sqrt{xy} - x}{x - 2x^2\sqrt{xy}}$$

Inverse Trigonometric Functions

1) Let $y = \sin^{-1} x$, find y'

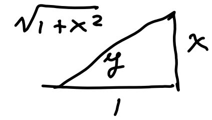


Sol: $\sin y = x$

$$\frac{d}{dx} \sin y = \frac{d}{dx}(x) \Rightarrow (\cos y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

2) Let $y = \tan^{-1} x$, find y'



Sol. $\tan y = x$

$$\frac{d}{dx} \tan y = \frac{d}{dx}(x) \Rightarrow \sec^2 y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+x^2}$$

Basic Formulas:

$$\frac{d}{dx} \sin^{-1} u = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$$

$$\frac{d}{dx} \tan^{-1} u = \frac{1}{1+u^2} \frac{du}{dx}$$

$$\frac{d}{dx} \sec^{-1} u = \frac{1}{u\sqrt{u^2-1}} \frac{du}{dx}$$

Examples

1) $y = \sin^{-1}(x^2)$ $\quad y' = \frac{1}{\sqrt{1-x^4}} \frac{d}{dx} x^2 = \frac{2x}{\sqrt{1-x^4}}$

2) $y = \tan^{-1} 5x$ $\quad y' = \frac{1}{1+25x^2} \frac{d}{dx}(5x) = \frac{5}{1+25x^2}$

3) $y = \sin^{-1}(e^x)$ $\quad y' = \frac{1}{\sqrt{1+e^{2x}}} \frac{d}{dx} e^x = \frac{e^x}{\sqrt{1+e^{2x}}}$

4) $y = \tan^{-1}(\cos\theta)$ $\quad y' = \frac{1}{1+\cos^2\theta} \frac{d}{d\theta} \cos\theta = \frac{-\sin\theta}{1+\cos^2\theta}$

3.6c Implicit Differentiation (Cont.)

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5. $y = x\cos^{-1}x - \sqrt{1-x^2}$

$y' = \cos^{-1}x + x\dfrac{1}{-\sqrt{1-x^2}} - \dfrac{1}{2\sqrt{1-x^2}}(-2x) = \cos^{-1}x$

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6. $y = \tan^{-1}[x - \sqrt{1+x^2}]$

$y' = \dfrac{1}{1+(x-\sqrt{1+x^2})^2} \cdot \dfrac{d}{dx}(x - \sqrt{1+x^2})$

$= \dfrac{1}{1+(x-\sqrt{1+x^2})^2}\left[1 - \dfrac{1}{2\sqrt{1+x^2}} \cdot 2x\right]$

$= \dfrac{1}{1+[x-\sqrt{1+x^2}]^2}\left[1 - \dfrac{x}{\sqrt{1+x^2}}\right]$

Not much one can do to simplify

Higher Order Derivatives

3.7 Higher Order Derivatives

Notation

Let $y = f(x)$

First derivative: $y' = f'(x) = \frac{dy}{dx} = Df(x) = Dy$

Second derivative $y'' = f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D^2 f(x)$

If $x = f(t)$ represents position

$\frac{dx}{dt} = v(t)$ represents velocity

$\frac{d^2x}{dt^2} = a(t)$ represents acceleration.

$\frac{d^3x}{dt^2} = j(t)$ represents a "jerk".

Examples

1) $y = x^3 - 5x^2 + 2x + 7$

$y' = 3x^2 - 10x + 2$

$y'' = 6x - 10$

$y''' = 6$

$y^{iv} = 0$

2. $y = x^4$

$y' = 4x^3$

$y'' = (4)(3)x^2$

$y''' = (4)(3)(2)x^1$

$y^{iv} = (4)(3)(2)(1) = 4!$

3) $f(x) = x^n$

$f^{(n)}(x) = n!$

4) $f(x) = e^x$

$f^{(n)}(x) = e^x$

5) $f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{iv}(x) = \sin x$

$f^{v}(x) = \cos x$

$\vdots$

"Cycle" of period 4

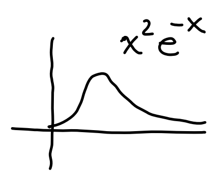
6) $y = x^2 e^{-x}$

$y' = x^2 e^{-x}(-1) + 2x e^{-x} = e^{-x}(-x^2 + 2x)$

$y'' = e^{-x}(-2x + 2) + (-x^2 + 2x)e^{-x}(-1)$

$= e^{-x}[-2x + 2 + x^2 - 2x]$

$= e^{-x}[x^2 - 4x + 2]$



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7. The position of a particle is given by $s(t) = t^3 - 12t^2 + 3t$, $t \geq 0$. Find

a) The acceleration after 3 seconds

$s'(t) = v(t) = 3t^2 - 24t + 3$

$s''(t) = a(t) = 6t - 24$ $\qquad a(3) = 6(3) - 24 = -6$

b) When is the particle speeding up?

$a(t) = 6t - 24 > 0$

$6t > 24$ $\qquad$ Speeding when $t > 4$

$t > 4$

51 **Harmonic Motion**

8. The position of a mass m attached to a spring with "stiffness" k is given by

$y(t) = A \sin \omega t$ $\qquad (A,\ \omega = \sqrt{\frac{k}{m}}$ const.)

a) Find the velocity and acceleration

$v(t) = y'(t) = A\omega \cos \omega t$

$a(t) = y''(t) = -A\omega^2 \sin \omega t$

b) Show that the acceleration is proportional to the displacement

$a = -A\omega^2 \sin \omega t$

$= -\omega^2 (A \sin \omega t)$

$a = -\omega^2 y$

c) Show that the force is given by $F = -ky$

$F = ma = -m\omega^2 y$

$= -m\left(\sqrt{\frac{k}{m}}\right)^2 y = -m\frac{k}{m} y \Rightarrow \boxed{F = -ky}$

3.8 Logarithmic Functions

Recall the definitions

1) $y = \ln x \iff e^y = x$ | 2) $y = \log_a x \iff a^y = x$

By implicit differentiation we get

$$\frac{d}{dx} e^y = \frac{d}{dx}(x)$$
$$e^y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$
$$\therefore \frac{d}{dx} \ln x = \frac{1}{x}$$
$$\Rightarrow \boxed{\frac{d}{dx} \ln u = \frac{1}{u}\frac{du}{dx}}$$

$$\ln a^y = \ln x$$
$$y \ln a = \ln x$$
$$(\ln a)\frac{dy}{dx} = \frac{d}{dx} \ln x$$
$$\frac{dy}{dx} = \frac{1}{\ln a}\frac{1}{x}$$
$$\Rightarrow \boxed{\frac{d}{dx} \log_a u = \frac{1}{u \ln a}\frac{du}{dx}}$$

I prefer to rewrite $\log_a x$ in terms of natural logarithms rather than memorizing one more formula

Examples

1. $y = \ln(\sec x)$
$$y' = \frac{1}{\sec x}\frac{d}{dx}(\sec x)$$
$$= \frac{1}{\sec x} \cdot \sec x \tan x$$
$$= \tan x$$
$$\Rightarrow \boxed{\frac{d}{dx}\ln(\sec x) = \tan x}$$

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2. $y = \ln(\sec x + \tan x)$
$$y' = \frac{1}{\sec x + \tan x} \cdot \frac{d}{dx}(\sec x + \tan x)$$
$$= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$
$$= \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x} = \boxed{\sec x}$$

The algebraic properties of $\ln x$ are often useful when computing derivatives

a) $\ln(uv) = \ln u + \ln v$
b) $\ln(u/v) = \ln u - \ln v$
c) $\ln u^v = v \ln u$
d) $\ln e^u = u$

Examples

1. $y = \ln\sqrt{x^3+9} = \ln(x^3+9)^{1/2} = \frac{1}{2}\ln(x^3+9)$
$y' = \frac{1}{2}\frac{1}{x^3+9}\cdot 3x^2 = \frac{3x^2}{2(x^3+9)}$

2. $y = \ln\sqrt{\frac{x-1}{x+1}} = \frac{1}{2}\ln\left(\frac{x-1}{x+1}\right) = \frac{1}{2}[\ln(x-1) - \ln(x+1)]$
$y' = \frac{1}{2}\left[\frac{1}{x-1} - \frac{1}{x+1}\right] = \frac{1}{2}\left[\frac{x+1-(x-1)}{x^2-1}\right] = \frac{1}{x^2-1}$

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3. $y = \ln[e^{-x} + xe^{-x}] = \ln[e^{-x}(1+x)]$
$= \ln e^{-x} + \ln(1+x) = -x + \ln(1+x)$
$y' = -1 + \frac{1}{1+x} = \frac{-(1+x)+1}{1+x} = \frac{-x}{1+x}$

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4. $y = \ln[x + \sqrt{x^2-1}]$
$y' = \frac{1}{x+\sqrt{x^2-1}}\cdot\frac{d}{dx}(x+\sqrt{x^2-1}) = \frac{1}{x+\sqrt{x^2-1}}\cdot\left(1+\frac{x}{\sqrt{x^2-1}}\right)$
$= \frac{1}{x+\sqrt{x^2-1}}\cdot\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}} = \frac{1}{\sqrt{x^2-1}}$ $= \cosh^{-1}x$ (3.9)

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5. $f(x) = \ln\frac{(2x+3)^3}{(3x-1)^4} = \ln(2x+3)^3 - \ln(3x-1)^4$
$= 3\ln(2x+3) - 4\ln(3x-1)$
$f'(x) = \frac{3}{2x+3}\cdot\frac{d}{dx}(2x+3) - \frac{4}{3x-1}\frac{d}{dx}(3x-1)$
$= \frac{6}{2x+3} - \frac{12}{3x-1}$

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6. $y = \sqrt[5]{\ln x}$ $y' = \frac{1}{5}(\ln x)^{-\frac{4}{5}}\cdot\frac{1}{x} = \frac{1}{5x\sqrt[5]{\ln^4 x}}$

7. $y = \ln \dfrac{e^x\sqrt{x+2}}{(x-3)^2(2x+1)^5}$

$y = \ln[e^x(x+2)^{1/2}] - \ln[(x-3)^2(2x+1)^5]$

$= \ln e^x + \ln(x+2)^{1/2} - \ln(x-3)^2 - \ln(2x+1)^5$

$= x + \frac{1}{2}\ln(x+2) - 2\ln(x-3) - 5\ln(2x+1)$

$y' = 1 + \dfrac{1}{2(x+2)} - \dfrac{2}{(x-3)} - \dfrac{10}{2x+1}$

5

8. $y = \dfrac{\ln u}{1+\ln 2u}$

$y' = \dfrac{(1+\ln 2u)(\ln u)' - \ln u(1+\ln 2u)'}{(1+\ln 2u)^2}$

$= \dfrac{(1+\ln 2u)(1/u) - \ln u(1/(2u))\cdot 2}{(1+\ln 2u)^2} \cdot \dfrac{u}{u}$

$= \dfrac{(1+\ln 2u) - \ln u}{(1+\ln 2u)^2} = \dfrac{1+\ln 2 + \ln u - \ln u}{(1+\ln 2u)^2}$

$= \dfrac{1+\ln 2}{(1+\ln 2u)^2}$

9. $y = \log_2(\sin x) \Rightarrow 2^y = \sin x \Rightarrow y\ln 2 = \ln(\sin x)$

$y'\ln 2 = \dfrac{1}{\sin x}\cdot\dfrac{d}{dx}(\sin x) = \dfrac{\cos x}{\sin x} = \cot x$

$y' = \dfrac{\cot x}{\ln 2}$

10. $y = 3^{4x+2}$ $\qquad \ln y = (4x+2)\ln 3$

$\frac{1}{y}y' = 4\ln 3$

$y' = (4\ln 3)3^{4x+2}$

This process is useful when we have situations of the following types

1) $y = a^{f(x)}$
2) $y = [f(x)]^{g(x)}$
3) $y =$ Expression with more than two products, quotients or radicals

Examples

1. $y = 2^{x^3}$

$\ln y = \ln 2^{x^3} = x^3 \ln 2$

$\frac{1}{y}\frac{dy}{dx} = 3x^2 \ln 2$

$y' = (3x^2 \ln 2)\, y = (3x^2 \ln 2)\, 2^{x^3}$

Note

$\frac{d}{dx} \ln y = \frac{1}{y}\frac{dy}{dx}$

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2. $y = \sqrt[4]{\frac{x^2+1}{x^2-1}}$ $\quad \ln y = \ln\left[\frac{x^2+1}{x^2-1}\right]^{1/4} = \frac{1}{4}\ln\left[\frac{x^2+1}{x^2-1}\right]$

$\ln y = \frac{1}{4}\left[\ln(x^2+1) - \ln(x^2-1)\right]$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{4}\left[\frac{2x}{x^2+1} - \frac{2x}{x^2-1}\right] = \frac{2x}{4}\left[\frac{x^2-1-(x^2+1)}{x^4-1}\right] = \frac{-x}{x^4-1}$

$\frac{dy}{dx} = -\frac{x}{x^4-1}\cdot y = \frac{-x}{x^4-1}\sqrt[4]{\frac{x^2+1}{x^2-1}}$

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3. $y = (\sin x)^x$ $\quad \ln y = x \ln(\sin x)$

$\frac{1}{y}\frac{dy}{dx} = \ln(\sin x) + x \cdot \frac{1}{\sin x}\cos x = \ln(\sin x) + x \cot x$

$\frac{dy}{dx} = [\ln(\sin x) + x \cot x](\sin x)^x$

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4. $y = (\ln x)^x$ $\quad \ln y = x \ln(\ln x)$

$\frac{1}{y}\frac{dy}{dx} = \ln(\ln x) + x \cdot \frac{1}{\ln x}\cdot\frac{d}{dx}\ln x = \ln(\ln x) + \frac{1}{\ln x}$

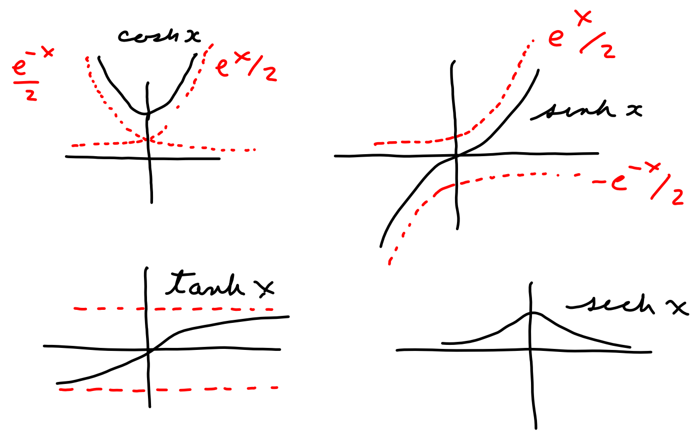
$\frac{dy}{dx} = \left[\ln(\ln x) + \frac{1}{\ln x}\right](\ln x)^x$

Hyperbolic Functions

3.9 Hyperbolic Functions

Definition

1. $\cosh x = \dfrac{e^{x} + e^{-x}}{2}$
2. $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$
3) $\tanh x = \dfrac{\sinh x}{\cosh x}$
4) $\operatorname{sech} x = \dfrac{1}{\cosh x}$



Basic Identity

$$\cosh^2 x - \sinh^2 x = \left[\frac{e^{x}+e^{-x}}{2}\right]^2 - \left[\frac{e^{x}-e^{-x}}{2}\right]^2$$

$$= \frac{1}{4}\left[e^{2x} + 2e^{x}e^{-x} + e^{-2x} - \left(e^{2x} - 2e^{x}e^{-x} + e^{-2x}\right)\right]$$

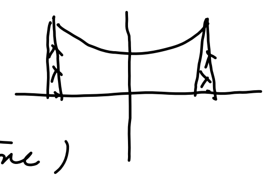
$$= \frac{1}{4}\left[4e^{x}e^{-x}\right] = \frac{1}{4}\cdot 4 = 1$$

$$\cosh^2 x - \sinh^2 x = 1$$

Applications:

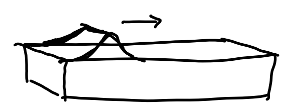
1. Hanging cable

   $y = c + a\cosh(x/a)$



2. Solitary waves (Solitons)

   $y = \operatorname{sech}^2 x$



   Single waves on long, narrow channels

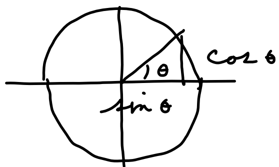
3.9

Ex: $\sinh(\ln 2) = \frac{1}{2}\left[e^{\ln 2} - e^{-\ln 2}\right]$

$= \frac{1}{2}\left[2 - \frac{1}{2}\right] = \frac{1}{2}\left(\frac{3}{2}\right) = \frac{3}{4}$

Geometry.

Circular Functions



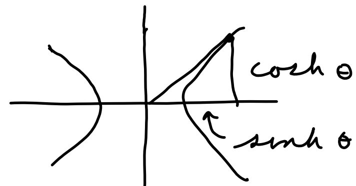
$x^2 + y^2 = 1$

$x = \cos\theta \qquad y = \sin\theta$

$\cos^2\theta + \sin^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

Hyperbolic Functions



$x^2 - y^2 = 1$

$x = \cosh\theta \qquad y = \sinh\theta$

$\cosh^2\theta - \sinh^2\theta = 1$

$1 - \tanh^2\theta = \operatorname{sech}^2\theta$

Derivatives:

1. $\frac{d}{dx}\cosh x = \frac{d}{dx}\frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \sinh x$

2. $\frac{d}{dx}\sinh x = \frac{d}{dx}\frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$

3) $\frac{d}{dx}\tanh x = \frac{d}{dx}\frac{\sinh x}{\cosh x} =$

$$= \frac{\cosh x(\sinh x)' - \sinh x(\cosh x)'}{\cosh^2 x}$$

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

4) $\frac{d}{dx}\operatorname{sech} x = \frac{d}{dx}[\cosh x]^{-1} = -(\cosh x)^{-2}\sinh x$

$$= -\frac{\sinh x}{\cosh^2 x} = -\frac{1}{\cosh x}\,\frac{\sinh x}{\cosh x}$$

$$= -\operatorname{sech} x \tanh x$$

Summary:

1. $\frac{d}{dx}\cosh u = \sinh u \frac{du}{dx}$
2. $\frac{d}{dx}\sinh u = \cosh u \frac{du}{dx}$
3. $\frac{d}{dx}\tanh u = \operatorname{sech}^2 u \frac{du}{dx}$
4. $\frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u \tanh u \frac{du}{dx}$

Examples

1. $y = \tanh 6x$
$y' = \operatorname{sech}^2 6x \cdot \frac{d}{dx}(6x)$
$= 6\operatorname{sech}^2 6x$

2. $y = \cosh x^4$
$y' = \sinh x^4 \cdot (x^4)'$
$= 4x^3 \sinh x^4$

3. $y = \sinh^2 x$
$y' = 2\sinh x \cdot \frac{d}{dx}(\sinh x)$
$= 2\sinh x \cosh x$

4. $y = \ln(\cosh x)$
$y' = \frac{\sinh x}{\cosh x} = \tanh x$

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5. $y = \sinh(\cosh x)$
$y' = \cosh(\cosh x)\cdot \frac{d}{dx}(\cosh x) = \sinh x \cdot \cosh(\cosh x)$

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6. $y = e^{\cosh 3x}$
$y' = e^{\cosh 3x}(\cosh 3x)' = (3\sinh 3x)\, e^{\cosh 3x}$

7. $y = \cosh(\ln x)$
$= \frac{1}{2}[e^{\ln x} + e^{-\ln x}] = \frac{1}{2}[x + \frac{1}{x}]$

$y' = \frac{1}{2}[1 - \frac{1}{x^2}] = \frac{x^2 - 1}{2x^2}$

3.9c Hyperbolic Functions (Cont.)

Inverse Hyperbolic Functions

Definition $y = \sinh^{-1} x \iff \sinh y = x$

$y = \tanh^{-1} x \iff \tanh y = x$

Derivatives

1. $y = \sinh^{-1} x \Rightarrow \sinh y = x$

$(\cosh y) \frac{dy}{dx} = \frac{dx}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{\cosh x} = \frac{1}{\sqrt{1 + \sinh^2 x}} = \frac{1}{\sqrt{1+x^2}}$$

2. $y = \tanh^{-1} x \Rightarrow \tanh y = x$

$(\operatorname{sech}^2 y) \frac{dy}{dx} = \frac{dx}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y} = \frac{1}{1 + \tanh^2 y} = \frac{1}{1-x^2}$$

Summary:

1. $\frac{d}{dx} \sinh^{-1} u = \frac{1}{\sqrt{1+u^2}} \frac{du}{dx}$
2. $\frac{d}{dx} \tanh^{-1} u \quad \frac{1}{1+u^2} \frac{du}{dx}$

Examples

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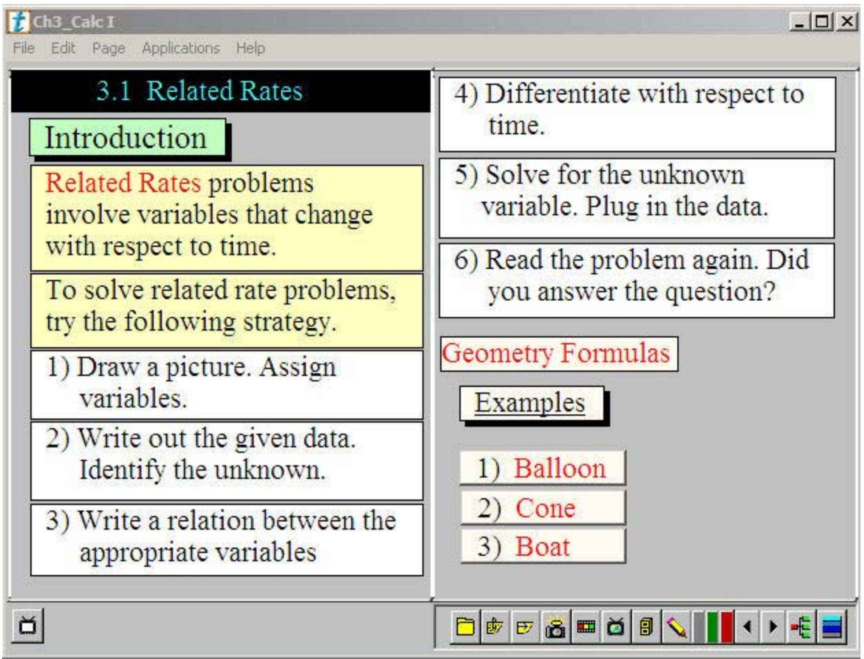
1. $y = \tanh^{-1} \sqrt{x} \qquad y' = \frac{1}{1-(\sqrt{x})^2} (\sqrt{x})' = \frac{1}{2\sqrt{x}(1-x)}$
2. $y = x \tanh^{-1} x + \ln \sqrt{1-x^2}$

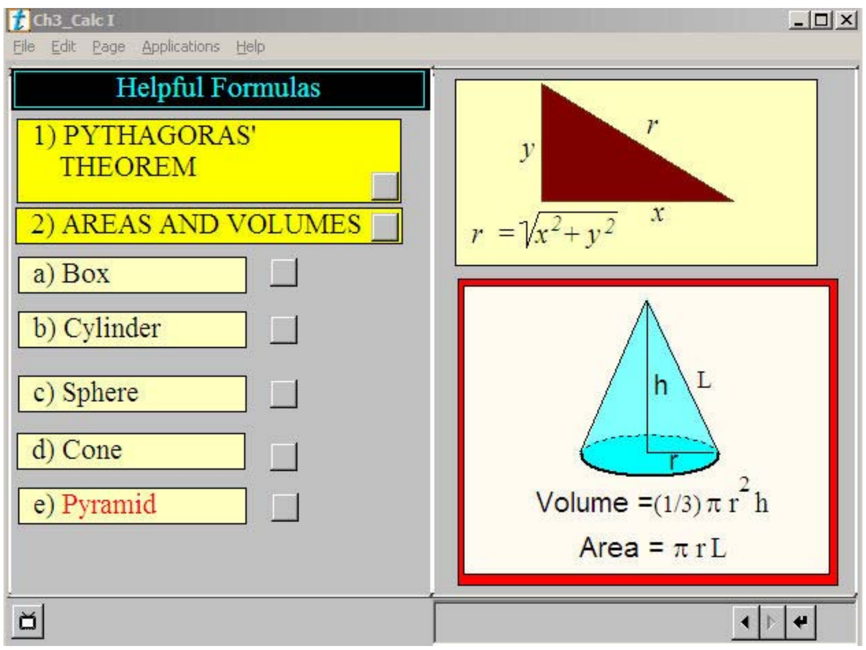
$y' = x \left(\frac{1}{1-x^2}\right) + \tanh^{-1} x + \frac{1}{2}\left(\frac{1}{1-x^2}\right)(-2x)$

$= \frac{x}{1-x^2} + \tanh^{-1} x - \frac{x}{1-x^2} = \tanh^{-1} x$

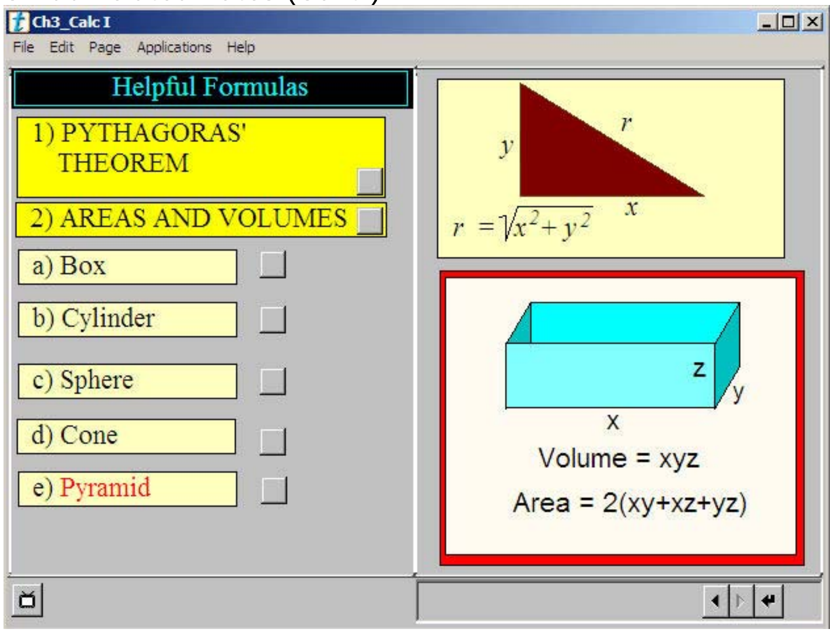
Related Rates

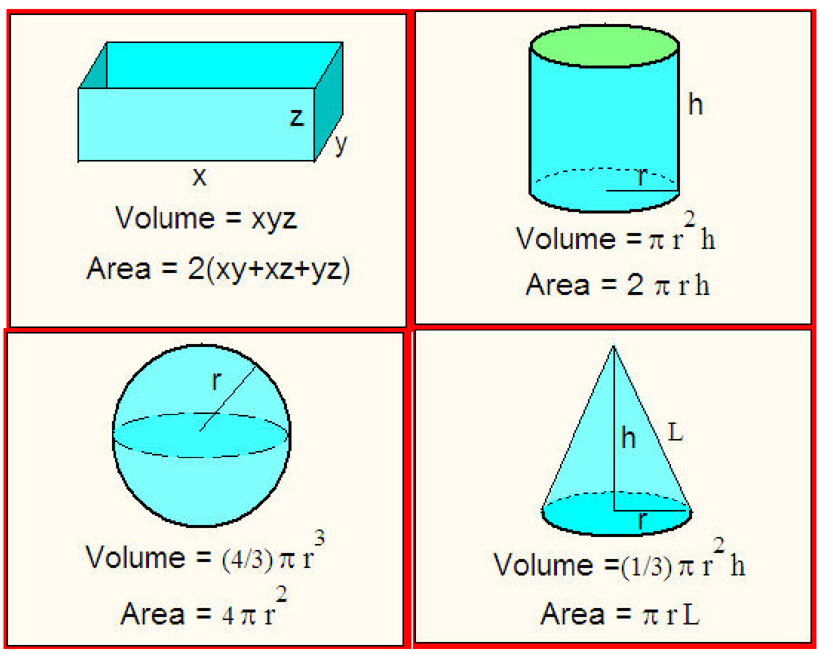
3.10 Related Rates



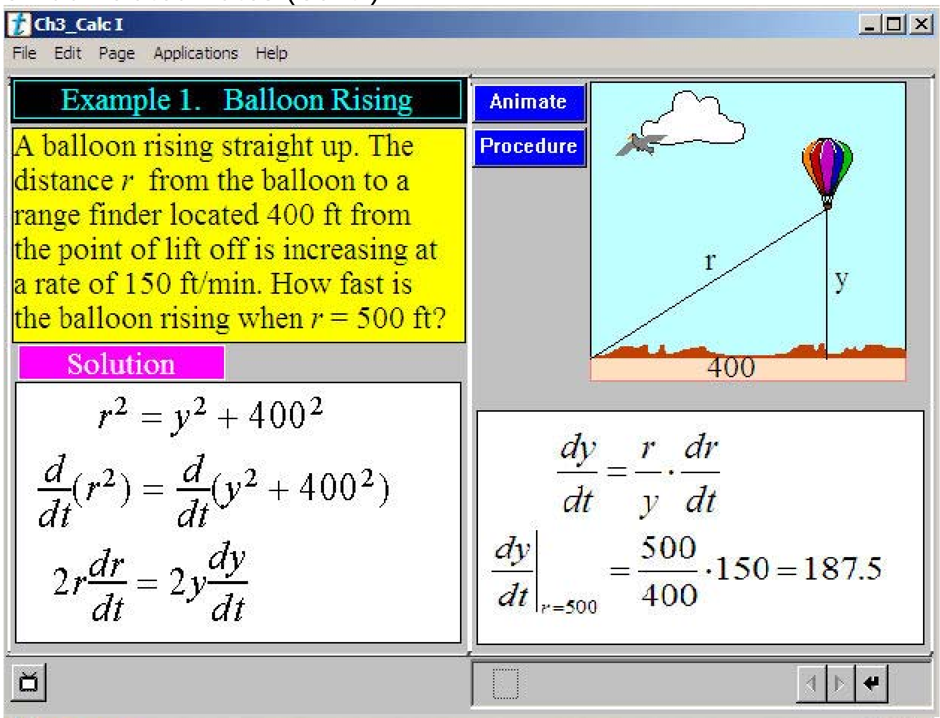


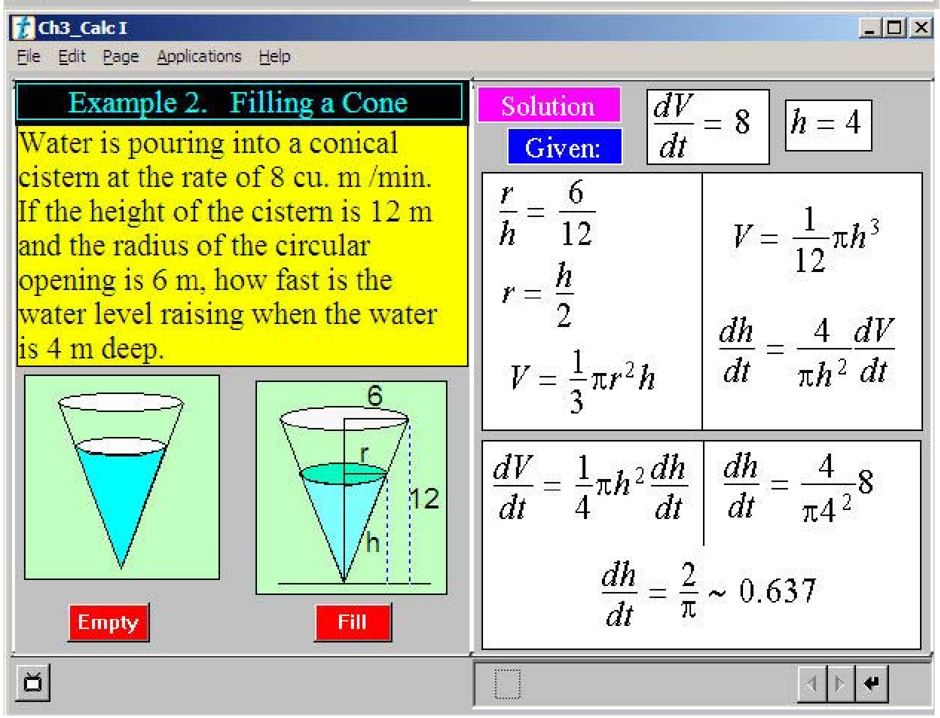
3.10b Related Rates (Cont.)

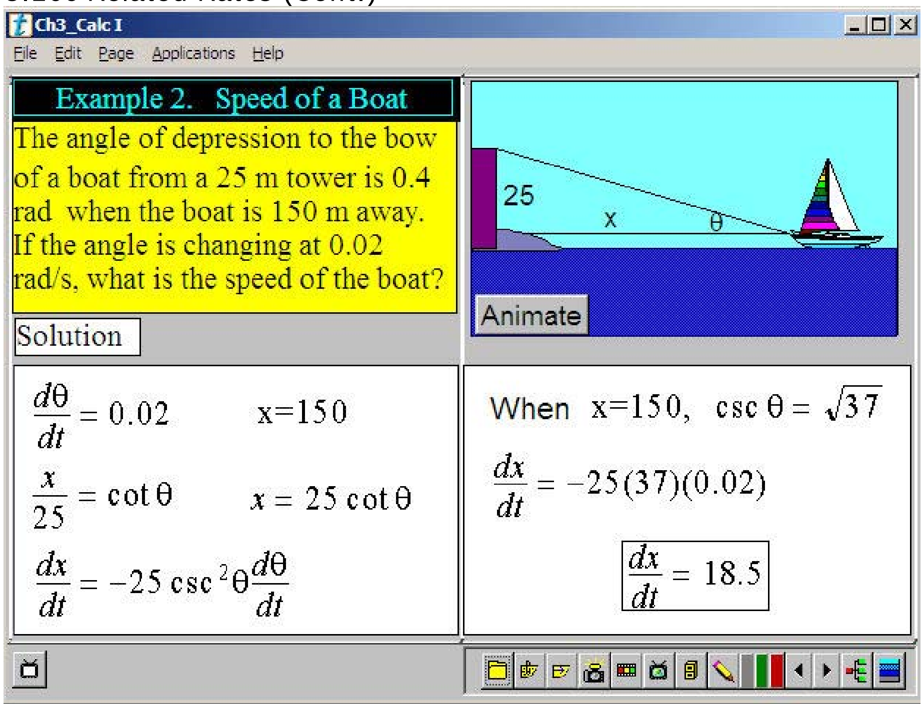




3.10b Related Rates (Cont.)



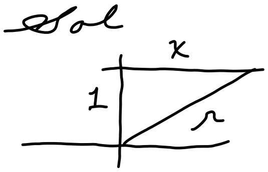




Examples

7 1. A plane flying horizontally at an altitude of 1 mile and a speed of 500 mph passes directly over a radar station. Find the rate at which the distance to the station is increasing when it is 2 miles away from the station.

Sol



$1 + x^2 = r^2$ $\quad \frac{dx}{dt} = 500$ $\quad x = 2$ $\quad \frac{dr}{dt} = ?$

$\frac{d}{dt}(1 + x^2) = \frac{d}{dt} r^2$

$2x\frac{dx}{dt} = 2r\frac{dr}{dt}$ $\quad \frac{dr}{dt} = \frac{x}{r}\frac{dx}{dt}$

$\left.\frac{dr}{dt}\right|_{x=2} = \frac{2}{\sqrt{5}} \cdot 500 \simeq 447.21$ mph

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2. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

Sol:

$S = 4\pi r^2 = 4\pi\left(\frac{D}{2}\right)^2$ $\quad D = 10, \quad \frac{dS}{dt} = -1$

$S = \pi D^2$

$\frac{dS}{dt} = 2\pi D \frac{dD}{dt}$ $\quad \left.\frac{dD}{dt}\right|_{D=10} = \left.\frac{1}{2\pi D}\frac{dS}{dt}\right|_{D=10} = \frac{-1}{20\pi}$

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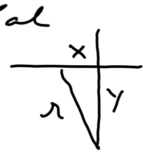
4. Two cars start moving from the same point. One travels south at 60 mph and the other travels east at 25 mph. At what rate is the distance between the cars increasing two hours later?

Sol



$\frac{dy}{dt} = 60$ $\quad$ At $t = 2$ $\quad y = 120$

$x = 50$ $\quad \frac{dr}{dt} = ?$

$\frac{dx}{dt} = 25$ $\quad \therefore\ r = 130$

$x^2 + y^2 = r^2 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt}$

$\left.\frac{dr}{dt}\right|_{t=2} = \left.\frac{1}{r}\left[x\frac{dx}{dt} + y\frac{dy}{dt}\right]\right|_{t=2} = \frac{1}{130}(50\cdot 25 + 120\cdot 60)$

$= 65$ mph

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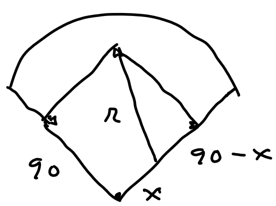
5. A baseball diamond is a square with side 90 ft. A batter hits the ball and runs towards first base at 24 ft/s. How fast is his distance from second base decreasing when he is halfway to first base?

Sol



$$(90-x)^2 + 90^2 = r^2$$

$$2(90-x)\left(-\frac{dx}{dt}\right) = 2r\,\frac{dr}{dt}$$

$x = 45$ ft

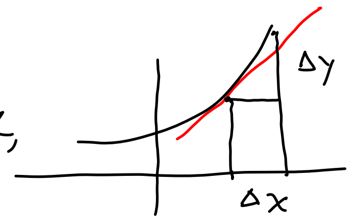
$\frac{dx}{dt} = 24$ ft/s

$$\left.\frac{dr}{dt}\right|_{x=45} = \left.-\frac{1}{r}(90-x)\frac{dx}{dt}\right|_{x=45}$$

$$= -\frac{1}{\sqrt{45^2+90^2}}\cdot(90-45)\cdot 24$$

$$= -\frac{1}{45\sqrt{1+2^2}}(45)(24) = -\frac{24}{\sqrt{5}}$$

3.11 Differentials



Let $y = f(x)$

As we have already learned, the derivative of y w.r.t x is given by the limit

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

If Δx is small then we can approximate $f'(x)$ by

$$f'(x) \simeq \frac{\Delta y}{\Delta x} \quad \Rightarrow \quad \Delta y \simeq f'(x)\,\Delta x$$

The smaller Δx the better the approximation. Thus, at the infinitesimal level, we are lead to the definition

$$dy = f'(x)\,dx$$

dx is called the differential of x

Differentials are useful for:

1. Approximations
2. Justification of the chain rule
3. Implicit differentiation
4. Linearization.

Examples

1. $y = x^2 \qquad dy = 2x\,dx$
2. $y = \sin x \qquad dy = \cos x\,dx$
3. $y = e^x \qquad dy = e^x\,dx$
4. $y = xe^x \qquad dy = (xe^x + e^x)\,dx$

Approximations

Ex: Find an approximation to $\sqrt{101}$ using differentials

Sol Let $y = \sqrt{x}$ $\quad x = 100 \quad \Delta x = 1 \quad y = 10$

$$\Delta y = \frac{1}{2\sqrt{x}} \Delta x = \frac{1}{2\sqrt{100}} \cdot 1 = \frac{1}{20} = 0.05$$

$$\sqrt{101} \simeq y + \Delta y = 10 + 0.05 \qquad \sqrt{101} \simeq 10.05$$

Chain Rule

Let $y = f(u)$, $u = g(x)$ $\quad$ Then $dy = f'(u)\,du$ $\quad$ (Def.)

$$\frac{dy}{dx} = f'(u)\frac{du}{dx} \qquad f'(u) = \frac{dy}{du}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

We can rewrite the derivative formulas of basic functions using differentials

Examples

1. $\frac{d}{dx} e^u = e^u \frac{du}{dx} \Leftrightarrow d(e^u) = e^u\,du$
2. $\frac{d}{dx} \sin u = \cos u \frac{du}{dx} \Leftrightarrow d(\sin u) = \cos u\,du$
3. $\frac{d}{dx} \ln u = \frac{1}{u}\frac{du}{dx} \Leftrightarrow d(\ln u) = \frac{1}{u}\,du$
4. $d(uv) = u\,dv + v\,du$ $\quad$ etc.

Implicit Differentiation

Examples

1. $x^2 + y^2 = 1$

$d(x^2 + y^2) = d(1)$

$2x\,dx + 2y\,dy = 0$

$2y\,dy = -2x\,dx$

$dy = -\frac{x}{y}\,dx$

$\frac{dy}{dx} = -\frac{x}{y}$

2. $x^3 + xy = 3$

$d(x^3 + xy) = 0$

$3x^2\,dx + x\,dy + y\,dx = 0$

$x\,dy = -(3x^2 + y)\,dx$

$dy = -\frac{1}{x}(3x^2 + y)\,dx$

$\frac{dy}{dx} = \frac{-(3x^2 + y)}{x}$

Linearization

The linearization of a function at a point is the equation of the tangent line at that point

Examples

1. Find the linearization of $x^3 + y^3 = 2$ at the point $(1,1)$

Sol: $d(x^3 + y^3) = 0$

$3x^2\,dx + 3y^2\,dy = 0$

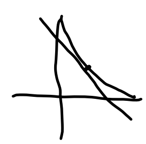
$y^2\,dy = -x^2\,dx$

$dy = -\frac{x^2}{y^2}\,dx$

At $(1,1)$

$\Delta y = y - 1$

$\Delta x = x - 1$



$y - 1 = -\left(\frac{1^2}{1^2}\right)(x-1) \Rightarrow y - 1 = -x + 1$

$y = x + 2$

Review

Differentiation Rules
Calculus I

General Formulas

1. $\frac{d}{dx}c = 0.$
2. $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}.$
3. $\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}v.$
4. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$
5. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$
6. $y = f(u),\ u = u(x),\ \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$

Basic Functions

1. $\frac{d}{dx}e^u = e^u\frac{du}{dx}.$
2. $\frac{d}{dx}\ln|u| = \frac{1}{u}\frac{du}{dx}.$
3. $\frac{d}{dx}\sin u = \cos u\frac{du}{dx}.$
4. $\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}.$
5. $\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}.$
6. $\frac{d}{dx}\sec u = \sec u\tan u\frac{du}{dx}.$
7. $\frac{d}{dx}\cot u = -\csc^2 u\frac{du}{dx}.$
8. $\frac{d}{dx}\csc u = -\csc u\cot u\frac{du}{dx}.$
9. $\frac{d}{dx}\sinh u = \cosh u\frac{du}{dx}.$
10. $\frac{d}{dx}\cosh u = \sinh u\frac{du}{dx}.$
11. $\frac{d}{dx}\tanh u = \operatorname{sech}^2 u\frac{du}{dx}.$
12. $\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}.$
13. $\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}.$
14. $\frac{d}{dx}\sec^{-1} u = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx}.$
15. $\frac{d}{dx}\sinh^{-1} u = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}.$
16. $\frac{d}{dx}\tanh^{-1} u = \frac{1}{1-u^2}\frac{du}{dx}.$