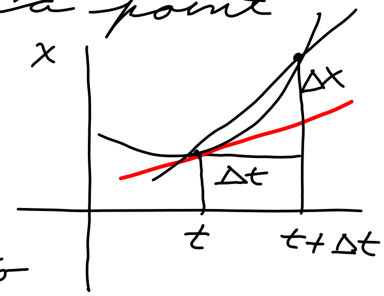
Tangents and Velocities

2.1 Tangents and Velocities

Let $x = f(t)$. We wish to find the slope of the tangent at a point
a) Graphically
b) Numerically



Idea

1. Pick a point nearby
2. Draw the secant line passing through the two points
3. Compute the slope of the secant

$$m_{sec} = \frac{\Delta x}{\Delta t}$$

4. As Δt approaches zero, the slope of the secant approaches the slope of the tangent (Drawn in red)

$$m_{tan} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

Interpretation

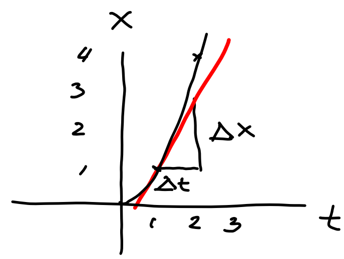
If $x = f(t)$ represents the position of a particle as a function of time then

1. Average Velocity $= \dfrac{\Delta x}{\Delta t}$

2. Instantaneous velocity $= \lim\limits_{\Delta t \to 0} \dfrac{\Delta x}{\Delta t}$

Example
Find the slope of the tangent to
$x = t^2$ at $t = 1$
a) Graphically
b) Numerically

Solution
a). Plot the function
- Draw the tangent
- Estimate the slope

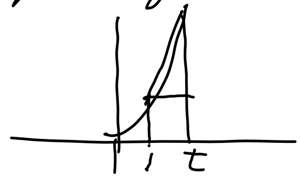


$$m_{sec} \simeq \frac{\Delta x}{\Delta t} \simeq 2$$

b)
- Pick a point t near 1 and write the formula for the slope of the secant

t	x
1	1
t	t^2

$$m_{sec} = \frac{\Delta x}{\Delta t} = \frac{t^2 - 1}{t - 1}$$



- Use a calculator to evaluate m_{sec} as $\Delta t \to 0$ (that is, as $t \to 1$)

t	m_{sec}
0.9	1.9
0.99	1.99
0.999	1.999

t	m_{sec}
1.1	2.1
1.01	2.01
1.001	2.001

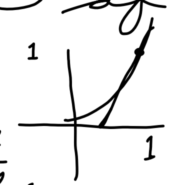
$$\Rightarrow \boxed{\lim_{t \to 1} \frac{t^2 - 1}{t - 1} = 2}$$

$$\lim_{t \to 1^-} \frac{t^2 - 1}{t - 1} = 2 \qquad \lim_{t \to 1^+} \frac{t^2 - 1}{t - 1} = 2$$

One sided limits

The equation of the tangent line to a curve at a point is the <u>linearization</u> of the curve at the point in question

Ex: Find the linearization of $x = t^2$ at $t = 1$



Sol: at $t = 1$, $x = 1$

We have just seen that at $t = 1$ the slope of the tangent is 2. Thus, all we need to do is find the of a straight line through the point $P(1, 1)$ with slope $m = 2$.

$$\frac{x-1}{t-1} = 2$$

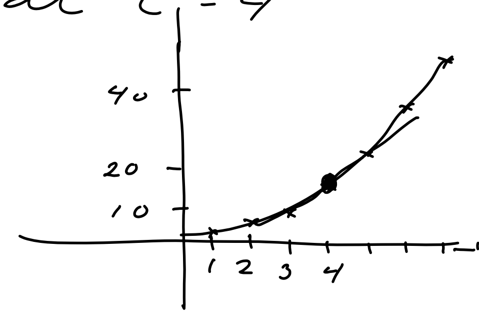
$$x - 1 = 2(t-1)$$
$$= 2t - 2$$
$$x = 2t - 1$$

At an infinitesimal level the equation of the parabola $x = t^2$ near $t = 1$ looks like a straight line. The equation of that line is what we call the linearization.

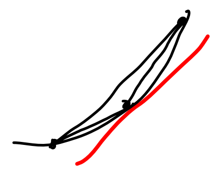
Functions Defined by Data

t	x
0	0
1	1.3
2	3.7
3	8.8
$\boxed{4}$	$\boxed{16.4}$
5	24.0
6	35.6
7	48.1
8	63.4

<u>Estimate</u> the instantane velocity at $t = 4$



$$\frac{\Delta x}{\Delta t} = \frac{24.0 - 8.8}{5 - 3} = 7.6$$



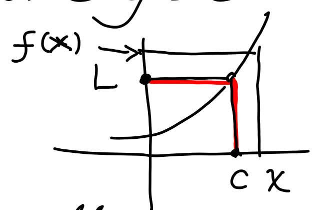
Better approximation by using the slope of the secant joining the point before ($t = 3$) and the point after ($t = 5$)

Limit of a Function

2.2 Limit of a function

Definition (Non-rigorous)

$\lim_{x \to c} f(x) = L$ if $f(x)$ is close to L whenever x is sufficiently close to c.



Equivalently:

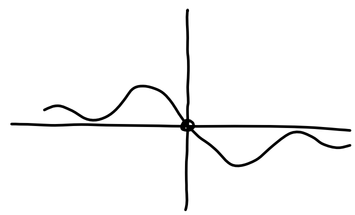
$|x - c|$ small $\Rightarrow |f(x) - L|$ small

Example

$$\lim_{x \to 0} \frac{\cos x - 1}{x}$$

x	$f(x)$
-0.1	0.0499
-0.01	0.005
-0.001	0.0005

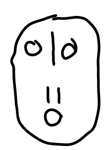
x	$f(x)$
0.1	-0.0499
0.01	-0.005
0.001	-0.0005



$|x - 0| < 0.001 \Rightarrow |f(x) - 0| < 0.0005$

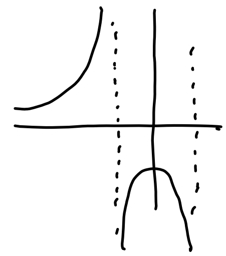
Guess: $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$

Warning!

If you write $\frac{0}{0} = 0$, you get  points

Infinite Limits

Example: Find $\lim_{x \to 1} \frac{1}{x^2 - 1}$



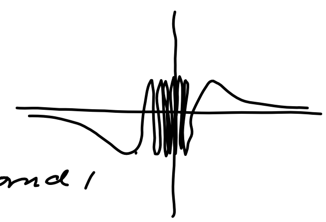
x	$f(x)$
0.9	-5.26
0.99	-50.25
0.999	-500.3

x	$f(x)$
1.1	4.76
1.01	49.75
1.001	499.75

$\lim_{x \to 1^-} \frac{1}{x^2-1} = -\infty \qquad \lim_{x \to 1^+} \frac{1}{x^2-1} = \infty$

$\lim_{x \to 1} \frac{1}{x^2-1}$ Does not exist (DNE)

Example Find $\lim_{x \to 0} \sin \frac{1}{x}$



As $x \to 0$ the function fluctuates rapidly between -1 and 1
The limit does not exist

Questions

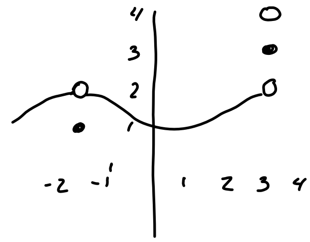
2.2 # 13

Sketch a function with

$\lim_{x \to 3^+} f(x) = 4$ $\lim_{x \to -2} f(x) = 2$

$\lim_{x \to 3^-} f(x) = 2$ $f(3) = 3$

$f(-2) = 1$

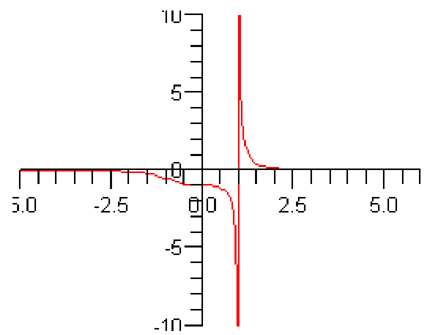


2.2 # 31

$\lim_{x \to 1^-} \frac{1}{x^3-1} = -\infty$

$\lim_{x \to 1^+} \frac{1}{x^3-1} = \infty$

$\lim_{x \to 1} \frac{1}{x^3-1}$ DNE



Computing Limits

2.3 Computing Limits

The Limit Theorem

Given $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$

Then a) $\lim_{x \to a} k f(x) = kL$

b) $\lim_{x \to a} (f+g)(x) = L + M$

c) $\lim_{x \to a} (f-g)(x) = L - M$

d) $\lim_{x \to a} (f \cdot g)(x) = L \cdot M$

e) $\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \frac{L}{M}$ $\quad M \neq 0$

f) $\lim_{x \to a} f(g(x)) = f(M)$

Example: Compute and justify all steps

$$\lim_{x \to 1} \left(\frac{x^3 - 1}{x - 1}\right) = \lim_{x \to 1} \frac{\cancel{(x-1)}(x^2 + x + 1)}{\cancel{(x-1)}}$$

$$= \lim_{x \to 1} (x^2 + x + 1)$$

$$= \lim_{x \to 1} x^2 + \lim_{x \to 1} x + \lim_{x \to 1} (1)$$

$$= \left(\lim_{x \to 1} x\right)^2 + \lim_{x \to 1} x + \lim_{x \to 1} (1)$$

$$= 1 + 1 + 1 = 3$$

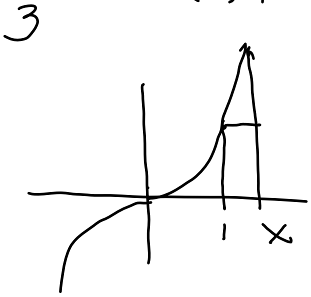
Note:

Let $y = f(x) = x^3$

x	$f(x)$
1	1
x	x^3

$m_{sec} = \frac{x^3 - 1}{x - 1}$

$m_{Tan} = \lim_{x \to 1} \frac{x^3 - 1}{x - 1}$



Example:
Compute.

Difference of Squares

$$\boxed{(a-b)(a+b) = a^2 - b^2}$$

$$\lim_{x \to 9} \left(\frac{\sqrt{x} - 3}{x - 9} \right)$$

$$= \lim_{x \to 9} \left(\frac{\sqrt{x} - 3}{x - 9} \right) \left(\frac{\sqrt{x} + 3}{\sqrt{x} + 3} \right)$$

$$= \lim_{x \to 9} \frac{\cancel{(x-9)}}{\cancel{(x-9)}(\sqrt{x} + 3)} = \lim_{x \to 9} \frac{1}{\sqrt{x} + 3}$$

$$= \frac{1}{\lim_{x \to 9} (\sqrt{x} + 3)} = \frac{1}{3 + 3} = \frac{1}{6}$$

2.3 # 25

$$\lim_{x \to -4} \left(\frac{\frac{1}{4} + \frac{1}{x}}{4 + x} \right) = \lim_{x \to -4} \frac{1}{(4+x)} \left(\frac{1}{4} + \frac{1}{x} \right)$$

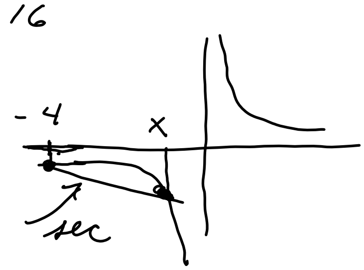
Common Denom.

$$= \lim_{x \to -4} \frac{1}{(4+x)} \left(\frac{x + 4}{4x} \right)$$

$$= \lim_{x \to -4} \frac{1}{4x} = -\frac{1}{16}$$

Note:
Let $y = f(x) = \frac{1}{x}$

x	$f(x)$
-4	$-\frac{1}{4}$
x	$\frac{1}{x}$



$$m_{sec} = \frac{\frac{1}{x} + \frac{1}{4}}{x + 4}$$

$$m_{tan} = \lim_{x \to -4} \left(\frac{\frac{1}{4} + \frac{1}{x}}{4 + x} \right)$$

Multiply by conjugate

2.3 # 31

$$\lim_{x\to 0} \frac{x}{\sqrt{1+3x}-1} = \lim_{x\to 0}\left[\frac{x}{\sqrt{1+3x}-1}\right]\left[\frac{\sqrt{1+3x}+1}{\sqrt{1+3x}+1}\right]$$

$$= \lim_{x\to 0} \frac{x\left[\sqrt{1+3x}+1\right]}{(1+3x)-1}$$

$$= \lim_{x\to 0} \frac{x\left[\sqrt{1+3x}+1\right]}{3x}$$

$$= \lim_{x\to 0} \frac{\sqrt{1+3x}+1}{3} = \frac{2}{3}$$

When computing limits of fractions which contain radicals it often helps to rationalize (or antirationalize) by multiplying by the conjugate.

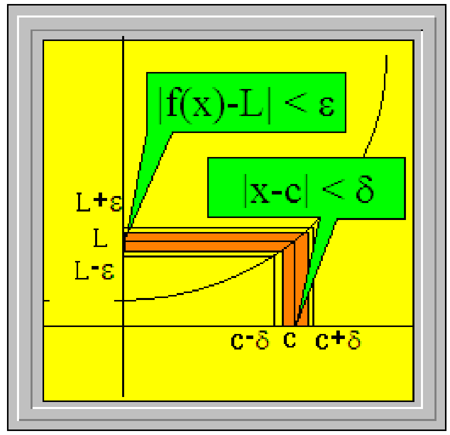
Rigorous Definition of Limit

2.4 Rigorous Definition of Limit

1) DEFINITION OF LIMIT

$$\lim_{x \to c} f(x) = L$$

The limit as x approaches c of a funtion f(x) is L, if for any number $\varepsilon > 0$, there exists a number $\delta > 0$, such that $|f(x)-L|<\varepsilon$, whenever $|x - c| < \delta$.



Idea

To prove that for a given function $f(x)$ the limit as x approaches c is equal to L one proceeds as follows

1. Let $\epsilon > 0$ be any (small) number
2. Choose a number $\delta > 0$ (depending on ϵ) such that

if $|x-c| < \delta$ (i.e if $|x-c|$ is small)
then $|f(x)-L| < \epsilon$ (i.e $|f(x)-L|$ is also small)

Since ϵ is an arbitrary number, this shows that we can make $f(x)$ arbitrarily close to L.

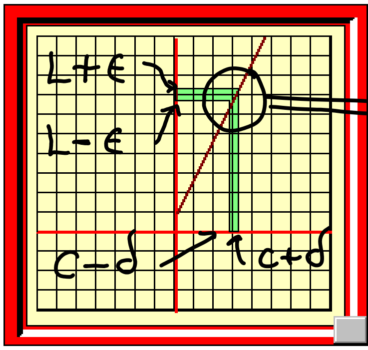
The difficult part of the proof is usually finding δ as a function of ϵ

For linear functions the problem is relatively easy.

2.4a Rigorous Definition of Limit (Cont.)

Example (Linear Function)

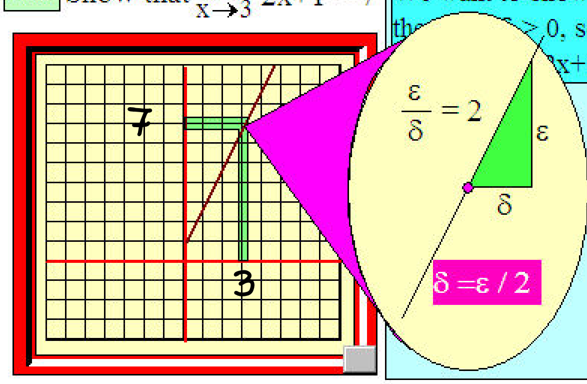
Ex1 Show that $\lim_{x\to 3} 2x+1 = 7$



We want to show that for any ε > 0, there is a δ > 0, so that |x-2| < δ implies that |(2x+1)-7|< ε

Zoom here to see how δ and ε are related

Ex1 Show that $\lim_{x\to 3} 2x+1 = 7$

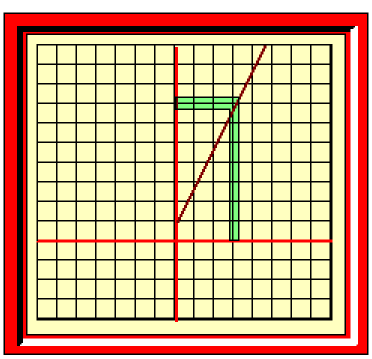


We want to show that for any ε > 0, th... > 0, so that |x-2| < δ ...x+1)-7|< ε

$\delta = \frac{\epsilon}{|m|}$

m : slope

Ex1 Show that $\lim_{x\to 3} 2x+1 = 7$



We want to show that for any ε > 0, there is a δ > 0, so that |x-2| < δ implies that |(2x+1)-7|< ε

Sol. Given any $\varepsilon > 0$, choose $\delta = \varepsilon/2$.

Assume: $|x-3| < \delta$, that is $|x-3| < \varepsilon / 2$.

Then $|f(x)-L| = |(2x+1)-7|$

$= |2x - 6| = 2|x - 3|$

$< 2 \cdot \varepsilon / 2 = \varepsilon$

So, $|f(x)-L| < \varepsilon$ whenever $|x-3| < \delta$.

Hence $\lim_{x\to 3} 2x+1 = 7$

3

Example

Show that $\lim_{x \to 4}(3x - 1) = 11$

Proof

Given $\epsilon > 0$ choose $\delta = \epsilon/3$

Assume that $|x-4| < \delta$, that is, $|x-4| < \frac{\epsilon}{3}$

Then,
$$\begin{aligned} |f(x) - L| &= |(3x-1) - 11| \\ &= |3x - 12| \\ &= 3|x-4| \\ &< 3 \cdot \frac{\epsilon}{3} \\ &< \epsilon \end{aligned}$$

We have shown that if $|x-4| < \delta$ then $|f(x) - L| < \epsilon$, hence $\lim_{x \to 4}(3x-1) = 11$

2.4 #9) Find a δ corresponding to $\epsilon = 0.1$ in the case $\lim_{x \to 1}(4 + x - 3x^3) = 2$

Sol Here $L = 2$, $2 + \epsilon = 2.1$

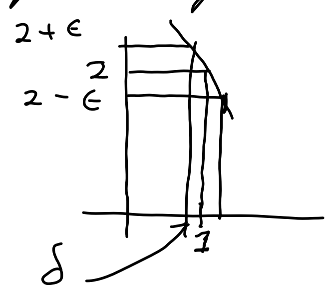
$c = 1$, $2 - \epsilon = 1.9$

With a graphing calculator find the x-coordinates corresponding to these numbers

x	y
0.988	1.9
1.011	2.1

$1 - 0.988 = 0.012$

$1.011 - 1 = 0.011$



$\delta = 0.010$ will do.

Definition $f(x)$ is continuous in an interval $[a,b] \subset \mathbb{R}$ if $f(x)$ is continuous at every point in the interval.

Theorem If $f(x)$ and $g(x)$ are continuous then so are

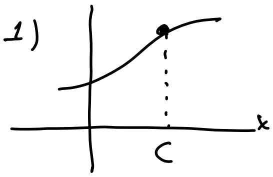
a) $kf(x)$ d) $(f \cdot g)(x)$
b) $(f+g)(x)$ e) $(f/g)(x)$ Provided $g(x) \neq 0$
c) $(f-g)(x)$ f) $f(g(x))$

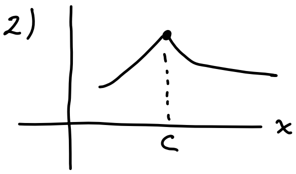
Corollary Polynomial functions are continuous for all $x \in \mathbb{R}$

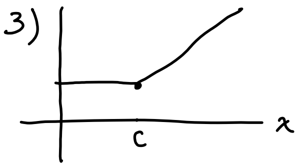
2.5 Continuity

Intuitive Examples

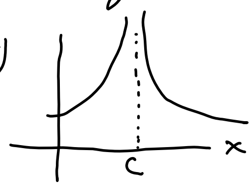
- The following are continuous at $x = c$

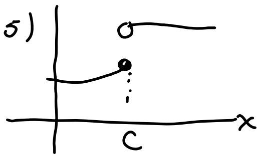
1) 

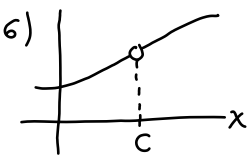
2) 

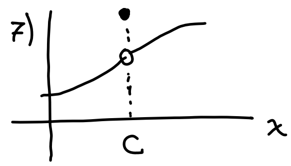
3) 

- The following are <u>not</u> continuous at $x = c$

4) 

5) 

6) 

7) 

<u>Definition</u>: A function $f(x)$ is continuous at $x = c$ if

a) $f(c)$ is defined

b) $\lim_{x \to c} f(x)$ exists

c) $\lim_{x \to c} f(x) = f(c)$

<u>Ex1:</u>) Is $f(x) = x^2 + 3x - 1$ continuous at $x = 1$?

<u>Sol</u>.

a) $f(1) = 1 + 3 - 1 = 3$. Defined ✓

b) $\lim_{x \to 1} f(x) = \lim_{x \to 1} (x^2 + 3x - 1) = 3$. Exists ✓

c) $\lim_{x \to 1} f(x) = f(1)$. Yes!

Ex 2) $f(x) = \dfrac{x^2-4}{x-2}$ continuous at $x=2$?

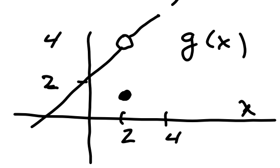
Ans: $f(2)$ not defined. No!

Ex 3) Is $g(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x \neq 2 \\ 1 & x = 2 \end{cases}$ cont. at $x=2$?

Sol: a) $g(2) = 1$ Defined!

b) $\lim\limits_{x\to 2} g(x) = \lim\limits_{x\to 2} \dfrac{x^2-4}{x-2} = \lim\limits_{x\to 2} \dfrac{(x+2)(x-2)}{(x-2)}$

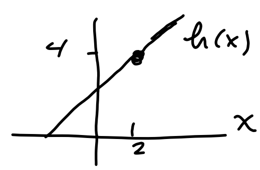
$= \lim\limits_{x\to 2} (x+2) = 4$



c) $\lim\limits_{x\to 2} g(x) \neq g(2)$. No

Ex 4) Is $h(x) = \begin{cases} \dfrac{x^2-4}{x-2} & x \neq 2 \\ 4 & x = 2 \end{cases}$ cont at $x = 2$?

Sol $\lim\limits_{x\to 2} h(x) = h(2) = 4$ Yes

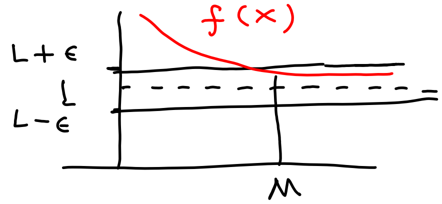


Note

A discontinuous function which can be redefined at a single point to make it continuous, is said to have a removable discontinuity at that point

$g(x)$ in example 3 has a removable discontinuity at $x = 2$

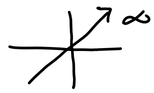
Definition: $\lim_{x \to \infty} f(x) = L$ if for every $\epsilon > 0$ there exists a number $M > 0$ such that $|f(x) - L| < \epsilon$ whenever $x > M$



Intuitively: $f(x)$ is close to L if x is large enough

Definition If $\lim_{x \to \infty} f(x) = L$ we call the line $y = L$ a horizontal asymptote

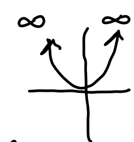
Ex 1 $\lim_{x \to \infty} x = \infty$



$\lim_{x \to -\infty} x = -\infty$

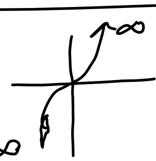
Ex 2 $\lim_{x \to \infty} x^2 = \infty$

$\lim_{x \to -\infty} x^2 = -\infty$



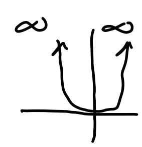
Ex 3 $\lim_{x \to \infty} x^3 = \infty$

$\lim_{x \to -\infty} x^3 = -\infty$



Ex 4 $\lim_{x \to \infty} x^4 = \infty$

$\lim_{x \to -\infty} x^4 = \infty$



Ex 5 Find $\lim_{x \to \infty} \frac{x^2+1}{x^2-1}$

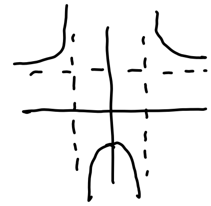
Sol 1 As x becomes arbitrarily large the $+1$ and -1 become insignificant so

$$\lim_{x \to \infty} \frac{x^2+1}{x^2-1} = 1$$

Sol 2

$$\lim_{x \to \infty} \frac{x^2+1}{x^2-1} = \lim_{x \to \infty} \left(\frac{x^2+1}{x^2-1}\right)\left(\frac{1/x^2}{1/x^2}\right) = $$

$$= \lim_{x \to \infty} \frac{1 + 1/x^2}{1 - 1/x^2} = \frac{1+0}{1-0} = 1$$



Note: $\boxed{\lim_{x \to \infty} \frac{1}{x} = 0}$

Let $P_n = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

$Q_m = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0$

be polynomials, and let

$$f(x) = \frac{P_n(x)}{Q_m(x)}.$$

Then

$$\lim_{x\to\infty} f(x) = \begin{cases} \frac{a_n}{b_m} & \text{if } n=m \\ \pm\infty & \text{if } n>m \\ 0 & \text{if } n<m \end{cases}$$

Ex 6 $\displaystyle\lim_{x\to\infty} \frac{3x^4+x^2}{5x^4-1} = \frac{3}{5}$ $(n=m=4)$

Ex 7 $\displaystyle\lim_{x\to\infty} \frac{2x^5+3x-1}{-x^4+8} = -\infty$

Ex 8 $\displaystyle\lim_{x\to\infty} \frac{x^3+2x+3}{x^7+5x^2+2} = 0$

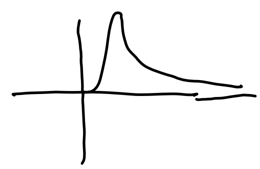
Rate of Growth of Functions.

Let $\prec$ denote "Grows slower than", Then

$$\ln x \prec x^n \prec e^x \prec x^x$$

Thus, for example, exponentials "dominate" polynomials, so

$$\lim_{x\to\infty} \frac{x^{100}}{e^x} = 0$$



2.6 # 23 $\lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right)$

$$= \lim_{x\to\infty}\left(\sqrt{9x^2+x}-3x\right)\frac{\left(\sqrt{9x^2+x}+3x\right)}{\left(\sqrt{9x^2+x}+3x\right)}$$

$$= \lim_{x\to\infty}\frac{9x^2+x-9x^2}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty}\frac{x}{\sqrt{9x^2+x}+3x}$$

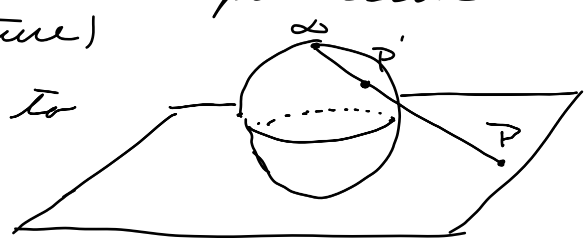
$$= \lim_{x\to\infty}\frac{x}{\sqrt{x^2(9+\frac{1}{x})}+3x} = \lim_{x\to\infty}\frac{x}{x\sqrt{9+\frac{1}{x}}+3x}$$

$$= \lim_{x\to\infty}\frac{1}{\sqrt{9+\frac{1}{x}}+3} = \frac{1}{\sqrt{9+0}+3} = \frac{1}{6}$$

Lesson: When working with limits involving radicals, rationalizing (or anti-rationalizing) often helps.

Stereographic Projection

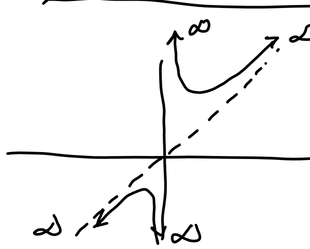
There is a 1-1 correspondence between points in the plane and points in a sphere. (See picture)



Infinity corresponds to the north pole

Rational Functions "going" to ∞ must "come back"

Ex $f(x) = x + \frac{1}{x}$



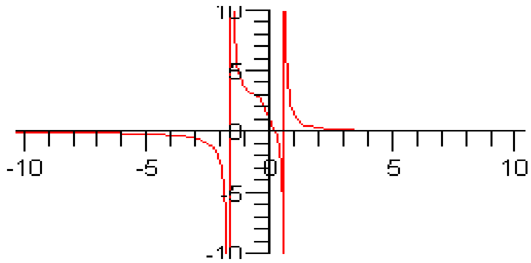
Example Find $\lim_{x\to\infty} \frac{x^3+5x-1}{2x^4+3x^3-1}$

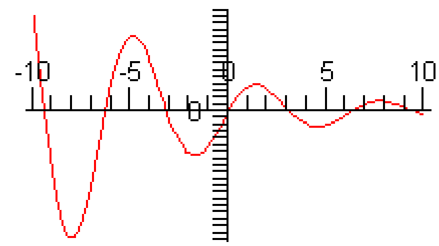
Sol 1. For large x the function is dominated by leading powers. Thus, asymptotically the function approaches the behaviour of $x^3/2x^4$ which goes to 0

$$\lim_{x\to\infty} \frac{x^3+5x-1}{2x^4+3x^3-1} = \lim_{x\to\infty}\left[\frac{x^3+5x-1}{2x^4+3x^3-1}\right]\frac{1/x^4}{1/x^4}$$

$$= \lim_{x\to\infty}\left[\frac{\frac{1}{x}+\frac{5}{x^3}-\frac{1}{x^4}}{2+\frac{3}{x}-\frac{1}{x^4}}\right] = \frac{0+0-0}{2+0-0} = 0$$

The line $y=0$ is a horizontal asymptote.





Example: Find $\lim_{x\to\infty} e^{-\frac{x}{6}}\sin x$

Sol $\lim_{x\to\infty} e^{-x/6}\sin x = \lim_{x\to\infty} \frac{\sin x}{e^{x/2}}$

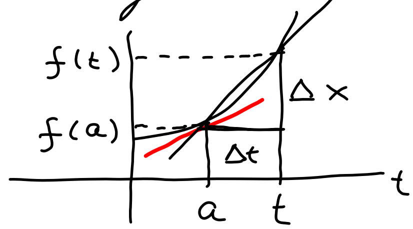
Since $|\sin x| \le 1$ while the denominator gets arbitrarily large as $x\to\infty$, the limit is 0

$$\lim_{x\to\infty} e^{-\frac{x}{6}}\sin x = 0$$

2.7 Velocities and Tangents

Let $x = f(t)$ denote the position of a particle as a function of time.
We define the velocity $v(a)$ at a point $x = a$ by the slope of the tangent at a

$$m_{sec} = v_{Ave} = \frac{\Delta x}{\Delta t} = \frac{f(t)-f(a)}{t-a}$$



As $\Delta t \to 0$ $m_{sec} \to m_{tan}$

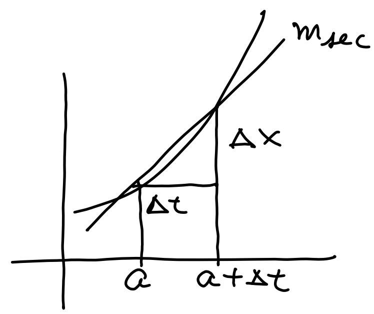
1 $$\boxed{v(a) = \lim_{t \to a} \frac{f(t)-f(a)}{t-a}}$$

This formula gives the instantaneous velocity for any fixed value $t = a$

The velocity $v(a)$ at a is also called the derivative of $f(t)$ at $t = a$ and it is often denoted by

$$f'(a) = v(a)$$

Alternative definition



$$m_{sec} = v_{Ave} = \frac{\Delta x}{\Delta t} = \frac{f(a+\Delta t)-f(a)}{\Delta t}$$

$$v(a) = f'(a) = \lim_{\Delta t \to 0} \frac{f(a+\Delta t)-f(a)}{\Delta t}$$

In the last formula we can replace a by t and find the velocity as a function of t!

2 $$\boxed{v(t) = f'(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t)-f(t)}{\Delta t}}$$

Note 2 $\Rightarrow$ 1 by setting $t = a$, $\Delta t = t - a$

Example: Let $x = t^2$. Find the velocity at $t=a$

Sol 1 $v(a) = \lim_{t \to a} \frac{t^2 - a^2}{t-a}$

t	$f(t)$
a	a^2
t	t^2

$$= \lim_{t \to a} \frac{(t-a)(t+a)}{t-a}$$

$$= \lim_{t \to a} (t+a)$$

$$= 2a$$

Sol 2 $v(t) = \lim_{\Delta t \to 0} \frac{(t+\Delta t)^2 - t^2}{\Delta t}$

$$= \lim_{\Delta t \to 0} \frac{t^2 + 2t\Delta t + \Delta t^2 - t^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{2t\Delta t + \Delta t^2}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\Delta t(2t + \Delta t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} (2t + \Delta t) \quad \leftarrow \text{Set } t = a$$

$$= 2t \qquad \Rightarrow \quad v(a) = 2a$$

The first formula usually works a little better when one only needs the velocity at one point

2.8 Derivatives

<u>Definition</u>: Let $y=f(x)$. The derivative of f with respect to x at $x=a$ is given by

$$f'(a)=\lim_{\Delta x\to 0}\frac{f(a+\Delta x)-f(a)}{\Delta x},$$

provided this limit exists

<u>Note</u>: If $\Delta x = x-a$ then $\Delta x\to 0$ is equivalent to $x\to a$ and the derivative at $x=a$ can be written as

$$f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

<u>Note</u>: $f'(a)$ represents the rate of change of $f(x)$ at $x=a$

<u>Example</u>: Find the equation of the tangent line to $y=3x^2-5x$ at the point $(2,2)$

<u>Sol</u>:

$$f'(2)=\lim_{\Delta x\to 0}\frac{f(2+\Delta x)-f(2)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[3(2+\Delta x)^2-5(2+\Delta x)-(12-10)\right]$$

$$=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[3(4+4\Delta x+\Delta x^2)-10-5\Delta x-2\right]$$

$$=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[12+12\Delta x+3\Delta x^2-10-5\Delta x-2\right]$$

$$=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[7\Delta x+3\Delta x^2\right]$$

$$=\lim_{\Delta x\to 0}(7+3\Delta x)=7$$

Tgt: $\frac{y-2}{x-2}=7$

$y-2=7x-14 \Rightarrow y=7x-16$

Example: Find $f'(a)$ of $f(x) = \dfrac{x}{2x-1}$

Sol: $f'(a) = \lim\limits_{\Delta x \to 0} \dfrac{1}{\Delta x}\left[\dfrac{(a+\Delta x)}{2(a+\Delta x)-1} - \dfrac{a}{2a-1}\right]$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{(a+\Delta x)(2a-1) - a[2(a+\Delta x)-1]}{[2(a+\Delta x)-1][2a-1]}\right]$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{2a^2 - a + 2a\Delta x - \Delta x - a(2a+2\Delta x-1)}{[2(a+\Delta x)-1]\cdot[2a-1]}\right]$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{\cancel{2a^2} - \cancel{a} + \cancel{2a\Delta x} - \Delta x - \cancel{2a^2} - \cancel{2a\Delta x} + \cancel{a}}{(2a+2\Delta x-1)(2a-1)}\right]$$

$$= \lim_{\Delta x \to 0} \frac{-1}{(2a+2\Delta x-1)(2a-1)} = \frac{-1}{(2a-1)^2}$$

Ex: Find $f'(a)$ of $f(x) = \dfrac{2}{\sqrt{3-x}}$

Sol: $f'(a) = \lim\limits_{\Delta x \to 0} \dfrac{1}{\Delta x}\left[\dfrac{2}{\sqrt{3-(a+\Delta x)}} - \dfrac{2}{\sqrt{3-a}}\right]$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{2\sqrt{3-a} - 2\sqrt{3-(a+\Delta x)}}{\sqrt{3-a-\Delta x}\,\sqrt{3-a}}\right]$$

$$= \lim_{\Delta x \to 0} \frac{2}{\Delta x}\left[\frac{\sqrt{3-a} - \sqrt{3-a-\Delta x}}{\sqrt{3-a-\Delta x}\,\sqrt{3-a}}\right]\left[\frac{\sqrt{3-a} + \sqrt{3-a-\Delta x}}{\sqrt{3-a} + \sqrt{3-a-\Delta x}}\right]$$

$$= \lim_{\Delta x \to 0} \frac{2}{\Delta x}\left[\frac{(\cancel{3}-\cancel{a}) - (\cancel{3}-\cancel{a}-\Delta x)}{\sqrt{3-a-\Delta x}\,\sqrt{3-a}}\right]\frac{1}{\sqrt{3-a} + \sqrt{3-a-\Delta x}}$$

$$= \lim_{\Delta x \to 0} \frac{2}{\cancel{\Delta x}}\,\frac{\cancel{\Delta x}}{\left[\sqrt{3-a-\Delta x}\,\sqrt{3-a}\right]\left[\sqrt{3-a} + \sqrt{3-a-\Delta x}\right]}$$

$$= \left[\frac{2}{\sqrt{3-a}\,\sqrt{3-a}}\right]\cdot\frac{1}{\sqrt{3-a} + \sqrt{3-a}}$$

$$= \left[\frac{2}{3-a}\right]\frac{1}{2\sqrt{3-a}} = \frac{1}{(3-a)^{3/2}}$$

Example: The position of a particle is given by $x(t) = t^2 - 6t - 5$. Find the velocity at time $t = 2$

Sol: $v(a) = \lim_{\Delta t \to 0} \frac{f(a+\Delta t) - f(a)}{\Delta t}$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[(a+\Delta t)^2 - 6(a+\Delta t) - 5 - (a^2 - 6a - 5)\right]$$

$$v(a) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[\cancel{a^2} + 2a\Delta t + \Delta t^2 - \cancel{6a} - 6\Delta t - \cancel{5} - \cancel{a^2} + \cancel{6a} + \cancel{5}\right]$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[2a\Delta t + \Delta t^2 - 6\Delta t\right]$$

$$= \lim_{\Delta t \to 0}\left[2a + \Delta t - 6\right] = 2a - 6$$

$$\Rightarrow v(2) = 2(2) - 6$$
$$= -2$$

2.9 Derivative as a Function

Definition A function $f(x)$ is differentiable at x if the limit

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

exists. In this case we call $f'(x)$ the derivative of $f(x)$ at x

It is often convenient to let $h = dx$. The formula then reads

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Example: Find $f'(x)$ if $f(x) = x + \sqrt{x}$

Sol:

$$f'(x) = \lim_{h \to 0} \frac{1}{h}\left[x + h + \sqrt{x+h} - (x + \sqrt{x}) \right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[h + \sqrt{x+h} - \sqrt{x} \right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[h + \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{\sqrt{x+h} + \sqrt{x}} \right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[h + \frac{(x+h) - x}{\sqrt{x+h} + \sqrt{x}} \right] = \lim_{h \to 0} \frac{1}{h}\left[h + \frac{h}{\sqrt{x+h} + \sqrt{x}} \right]$$

$$= \lim_{h \to 0} \left[1 + \frac{1}{\sqrt{x+h} + \sqrt{x}} \right]$$

$$= 1 + \frac{1}{2\sqrt{x}}$$

Example: Find $f'(x)$ of $f(x) = \dfrac{x+1}{x-1}$

Sol: $f'(x) = \lim\limits_{h \to 0} \dfrac{1}{h}\left[\dfrac{x+h+1}{x+h-1} - \dfrac{x+1}{x-1}\right]$

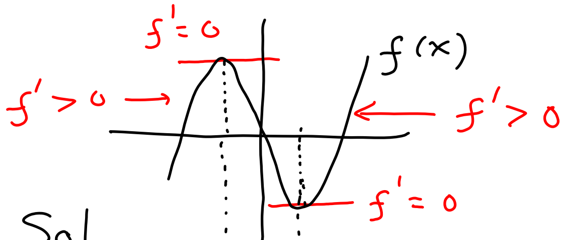
$$f'(x) = \lim_{h \to 0} \frac{1}{h}\left[\frac{(x-1)(x+h+1) - (x+1)(x+h-1)}{(x+h-1)(x-1)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{(\cancel{x^2} + \cancel{xh} + \cancel{x} - \cancel{x} - h - \cancel{1}) - (\cancel{x^2} + \cancel{xh} - x + \cancel{x} + h - \cancel{1})}{(x+h-1)(x-1)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{-2h}{(x+h-1)(x-1)}\right] = \lim_{h \to 0} \frac{-2}{(x+h-1)(x-1)}$$

$$= \frac{-2}{(x-1)^2}$$

Example Graph the derivative of $f(x)$



Sol